Optimal Stopping of a Diffusion with a Change Point *

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Abstract

This paper solves Bayes sequential optimal stopping and impulse control problems of a diffusion, whose drift term has an unobservable parameter with a change point. The value functions of the optimization and the control problems are characterized as viscosity solutions to non-stationary variational inequalities. Approximation schemes are proposed for the numerical computation of the value functions, thus also for the optimal stopping times and the optimal impulse controls.

Keywords and Phrases: Bayes sequential, optimal stopping, impulse control, parabolic variational inequalities, viscosity solutions.

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1 Introduction

Suppose the price evolution of a stock follows a geometric Brownian motion model, whose drift will change, at an unknown future time to an unknown level. An investor, who purchases a certain small number of shares at an initial time, can only observe the prices afterwards. Based on the observed prices and some reasonable *a priori* knowledge about the change in the drift, what is the best time to sell the shares, in order to maximize the profit on average? Under the same circumstance, what about the optimal discretely balanced buying-selling trading strategies with a larger number of shares?

At a more abstract level, this is the optimal stopping problem and the impulse control problem of a diffusion, whose drift term has an unobservable parameter with a change point. There have been routinely two common approaches to such problems. The conservative approach is the mini-max philosophy that optimizes in the worst case scenarios, formulated as a zero-sum game between a controller and a stopper by [10] Karatzas and Zamfirescu (2008). The approach employed by most, if not all, practitioners is to divide model calibration and decision making into two separate steps. Bayes sequential detection of change points and optimal stopping of diffusions are both very well developed fields with an extensive literature from the past decades. Among them, interested readers are referred to [16] Shiryaev (1969) and [7] Karatzas (2003) for sequential detection, and to [17] Shiryaev (1978) or Appendix D in [9] Karatzas and Shreve (1998) for optimal stopping problems.

This paper is an attempt at solving the optimal stopping and the impulse control problems in the Bayes sequential framework within one step, instead of conducting model calibration and decision making separately. This is facilitated by a change of measure, which hides the drift part of the diffusion. Under the new measure, an augmented state process is Markovian, one can therefore derive the variational inequalities satisfied by the value functions. The current values of the augmented state process provide all the information necessary for the decision making.

2 Optimal Stopping

2.1 The Model

Consider the canonical probability space $(\Omega, \mathbb{F}, \mathbb{P})$, which supports a one-dimensional standard Brownian motion $W(\cdot)$ with respect to its generated filtration $\mathscr{F}^W$. In this probability space, $\Omega = C[0,T]$ is the set of all continuous one-dimensional function on a finite deterministic time horizon $[0,T]$, $\mathbb{F} = \mathscr{B}(C[0,T])$ is the Borel sigma algebra, and $\mathbb{P}$ is the Wiener measure. The state process evolves according to the one-dimensional diffusion

$$dX(t) = b(t, X(t); \theta(t))dt + \sigma(t, X(t))dW(t), \tag{2.1}$$

for $0 \le t \le T$, with the initial value $X(0) = x_0 \in \mathbb{R}$. The unobservable parameter

$$\begin{aligned} \theta : [0,T] \times \Omega &\to \Theta, \\ (t,\omega) &\mapsto \theta(t,\omega) =: \theta(t) \end{aligned} \tag{2.2}$$

takes values in the parameter space $\Theta = \{\mu_0, \mu_1, \cdots, \mu_m\} \subset \mathbb{R}$. The parameter $\theta(\cdot)$ starts with initial value $\theta(0) = \mu_0$, and keeps this value until an unobservable time ρ of regime change. At the time ρ, the parameter $\theta(\cdot)$ changes to a new level U, a random variable whose value can be any one of the numbers $\mu_1, \mu_2, \cdots, \mu_m$, and remains at that level until the fixed finite terminal time $T > 0$. If the regime change does not occur by time T, then $\theta(\cdot)$ takes the value μ_0 throughout the interval $[0, T]$. That is to say,

$$\theta(t) = \begin{cases} \mu_0, \ 0 \le t < \rho \wedge T; \\ U, \ \rho \wedge T \le t \le T. \end{cases} \tag{2.3}$$

The change point ρ and the level U are two independent $\mathbb{F}$-measurable random variables, and the vector (ρ, U) is independent of the Brownian filtration $\mathscr{F}^W$. The independent random variables ρ and U are endowed with the prior distributions

$$\mathbb{P}(\rho > t) = e^{-\lambda t}, \ t \ge 0, \tag{2.4}$$

and

$$\mathbb{P}(U = \mu_j) = p_j, \ j = 1, 2, \cdots, m. \tag{2.5}$$

For any possible values $u \in \Theta$, the coefficients $b(\cdot, \cdot; u) : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $\sigma(\cdot, \cdot; u) : [0, T] \times \mathbb{R} \to \mathbb{R}$ are deterministic measurable functions satisfying the following locally Lipschitz and boundedness condition.

Assumption 2.1 *(1) For every compact subset $K_n \subset \mathbb{R}$, there exists a constant $C_n > 0$, such that*

$$|b(t, x^1; u) - b(t, x^2; u)| + |\sigma(t, x^1) - \sigma(t, x^2)| + \left| \frac{b(t, x^1; u)}{\sigma(t, x^1)} - \frac{b(t, x^2; u)}{\sigma(t, x^2)} \right| \le C_n |x^1 - x^2|, \tag{2.6}$$

for all (t, x^1), $(t, x^2) \in [0, T] \times K_n$, and for all $u \in \Theta$.
(2) There exists a constant $C > 0$, such that

$$\left| \frac{b(t, x; u)}{\sigma(t, x)} \right| \le C \tag{2.7}$$

holds for all $(t, x) \in [0, T] \times \mathbb{R}$, and for all $u \in \Theta$.
(3) There exists a constant $C > 0$, such that

$$|\sigma(t, x)| \le Cx \tag{2.8}$$

holds for all $(t, x) \in [0, T] \times \mathbb{R}$.

Under the local Lipschitz condition and linear growth condition, Assumption 2.1(1)(3), for b and σ, the stochastic differential equation (2.1) has a pathwise unique, strong solution $X(\cdot)$, whose generated filtration is denoted as $\mathscr{F} := \{\mathscr{F}_t\}_{0 \le t \le T}$. What is observable, in this context, is the process $X(\cdot)$ only. From the Bayes point of view, the prior distributions in (2.4), (2.5) for the independent random variables ρ, U, are the *a priori* knowledge about the parameter $\theta(\cdot)$ before observing the data process $X(\cdot)$.

We provide two commonly seen simple examples to illustrate the model (2.1).

Example 2.1 *(Brownian motion with drift uncertainty)*
The diffusion $X(\cdot)$ in (2.1) is a drifted Brownian motion

$$\begin{cases} dX(t) = & \theta(t)dt + dW(t); \\ X(0) = & 0. \end{cases} \tag{2.9}$$

The parameter $\theta(\cdot)$ is the drift with the initial value μ_0. The random variable U has the prior distribution

$$U = \begin{cases} \mu_+, \text{ with probability } p_+; \\ \mu_-, \text{ with probability } p_- = 1 - p_+, \end{cases} \tag{2.10}$$

and ρ has an exponential λ prior distribution as in (2.4).

Example 2.2 *(Geometric Brownian motion with drift uncertainty)*
The diffusion $X(\cdot)$ in (2.1) is a geometric Brownian motion

$$\begin{cases} dX(t) = & X(t)\theta(t)dt + X(t)\sigma dW(t); \\ X(0) = & x_0. \end{cases} \tag{2.11}$$

In this example, the volatility σ is a deterministic positive number. The parameter $\theta(\cdot)$ with the initial value μ_0 is the percentage drift of the Geometric Brownian motion. The random variables ρ and U have prior distributions (2.4) and (2.5).

2.2 General Theory

This subsection studies an optimal stopping problem of the diffusion specified by (2.1). Since the parameter $\theta(\cdot)$ and Brownian noise $W(\cdot)$ are unobservable, while the diffusion $X(\cdot)$ itself provides the only observations, we look for a stopping time τ^* with respect to the filtration $\mathscr{F}$, the information generated by $X(\cdot)$, to achieve the supremum

$$\sup_{\tau \in \mathscr{T}} \mathbb{E}\left[\int_0^\tau h(X(s))ds + \xi(X(\tau))\right], \tag{2.12}$$

where $\mathscr{T}$ is the collection of all $\mathscr{F}$-stopping times with values between 0 and T. Similarly, let $\mathscr{T}_t$ be the collection of all $\mathscr{F}$-stopping times with values between t and T. We call

$$V(t) := \sup_{\tau \in \mathscr{T}_t} \mathbb{E}\left[\int_t^\tau h(X(s))ds + \xi(X(\tau)) \middle| \mathscr{F}_t\right], \ 0 \le t \le T \tag{2.13}$$

the value process of the optimization problem (2.12). The rewards ξ and $h : \mathbb{R} \to \mathbb{R}$ are deterministic measurable functions satisfying the following conditions.

Assumption 2.1 *(continued)*
(4) The function $\xi(\cdot)$ is twice continuously differentiable, with first and second order derivatives denoted as $\xi'(\cdot)$ and $\xi''(\cdot)$.
(5) The functions $h(\cdot)$, $\xi(\cdot)$, $\xi'(\cdot)$ and $\xi''(\cdot)$ are locally Lipschitz and have polynomial growth.

From classical results on optimal stopping problems of continuous processes (c.f. Appendix D in [9] Karatzas and Shreve (1998)), the optimal stopping time for (2.12) has an expression

$$\tau^* = \inf \{ 0 \le t \le T \, | V(t) \le \xi(X(t)) \} . \tag{2.14}$$

The value process $V(\cdot)$, and thus the expression (2.14), are hard to compute directly in the $\mathbb{P}$-expectation. Instead, we shall consider the first hitting time τ^* under a so-called "reference probability measure" $\mathbb{P}_0$ for the diffusion $X(\cdot)$.

Let

$$\mathscr{G} = \{\mathscr{G}_t\}_{0 \le t \le T} = \sigma \{\rho, U, X(s); 0 \le s \le t\}_{0 \le t \le T} \tag{2.15}$$

be the filtration generated by ρ, U and $X(\cdot)$; this is larger than the filtration $\mathscr{F}$. Construct a probability measure $\mathbb{P}_0$, under which the $\mathscr{F}$-adapted process

$$W^0(t) := \int_0^t \sigma^{-1}(s, X(s)) dX(s), \; 0 \le t \le T \tag{2.16}$$

is a standard Brownian motion. With this standard $\mathbb{P}_0$-Brownian motion $W^0(\cdot)$, the pathwise unique, strong solution $X(\cdot)$ to the SDE (2.1) can be expressed alternatively as

$$X(t) = x_0 + \int_0^t \sigma(s, X(s)) dW^0(s), \; 0 \le t \le T. \tag{2.17}$$

Under Assumption 2.1(1)(3), the equation (2.17), as an SDE, has a pathwise unique, strong solution. The process $X(\cdot)$ is a local $(\mathbb{P}_0, \mathscr{F})$-martingale having the instantaneous quadratic variation $\sigma^2(\cdot, X(\cdot))$. In the probability space $(\Omega, \mathscr{G}, \mathbb{P}_0)$, the random variables ρ and U are independent, and the random vector (ρ, U) is independent of the process $X(\cdot)$. We assign the random variables ρ and U the same $\mathbb{P}_0$-prior distributions mandated by (2.4) and (2.5).

The Radon-Nikodym derivative process is the $\mathscr{G}$-adapted process defined as

$$Z(t) = \exp \left\{ \int_0^t \frac{b(s, X(s); \theta(s))}{\sigma^2(s, X(s))} dX(s) - \frac{1}{2} \int_0^t \frac{b^2(s, X(s); \theta(s))}{\sigma^2(s, X(s))} ds \right\}, \; 0 \le t \le T; \tag{2.18}$$

whereas, for every number $u \in \Theta$, we introduce the $\mathscr{F}$-adapted likelihood ratio process

$$L(t; u) = \exp \left\{ \int_0^t \frac{b(s, X(s); u)}{\sigma^2(s, X(s))} dX(s) - \frac{1}{2} \int_0^t \frac{b^2(s, X(s); u)}{\sigma^2(s, X(s))} ds \right\}, \; 0 \le t \le T. \tag{2.19}$$

From the expression (2.3) for $\theta(\cdot)$, the Radon-Nikodym derivative $Z(\cdot)$ can be written, in terms of the likelihood ratio process $L(\cdot; u)$ and the random vector (ρ, U), as

$$Z(t) = L(\rho; \mu_0) \left(\sum_{j=1}^m \mathbb{1}_{\{U = \mu_j\}} \frac{L(t; \mu_j)}{L(\rho; \mu_j)} \right) \mathbb{1}_{\{\rho < t\}} + L(t; \mu_0) \mathbb{1}_{\{\rho \ge t\}}, \; 0 \le t \le T. \tag{2.20}$$

There exists then a probability measure $\tilde{\mathbb{P}}$ equivalent to $\mathbb{P}_0$, satisfying

$$\left. \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}_0} \right|_{\mathscr{G}_t} = Z(t), \; 0 \le t \le T. \tag{2.21}$$

Under the probability measure $\tilde{\mathbb{P}}$, the random variables ρ and U are still independent and retain the prior distributions of (2.4) and (2.5). By a generalization of Girsanov theorem to local martingales in [15] Van Schuppen and Wong (1974), the process

$$\left\{X(t) - \int_0^t b(s, X(s); \theta(s))ds\right\}_{0\le t\le T} \tag{2.22}$$

is a local $(\tilde{\mathbb{P}}, \mathscr{G})$-martingale, also having the instantaneous quadratic variation $\sigma^2(\cdot, X(\cdot))$. The process $\tilde{W}(\cdot)$ defined as

$$\tilde{W}(t) := \int_0^t \sigma^{-1}(s, X(s))dX(s) - \int_0^t \sigma^{-1}(s, X(s))b(s, X(s); \theta(s))ds,\ 0 \le t \le T \tag{2.23}$$

is a continuous local $(\tilde{\mathbb{P}}, \mathscr{G})$-martingale with quadratic variation t, thus a standard $\tilde{\mathbb{P}}$-Brownian motion. The process $X(\cdot)$ satisfies the SDE (2.1) with the $\mathbb{P}$-Brownian motion $W(\cdot)$ replaced by the $\tilde{\mathbb{P}}$-Brownian motion $\tilde{W}(\cdot)$, thus having the same probabilistic properties under the probability measures $\mathbb{P}$ and $\tilde{\mathbb{P}}$. This means that the probability measures $\mathbb{P}$ and $\tilde{\mathbb{P}}$ coincide on the filtration $\mathscr{G}$.

By the Bayes rule, we may now compute (2.12) under the reference probability measure $\mathbb{P}_0$ as

$$\begin{aligned}
V(0) &= \sup_{\tau\in\mathscr{T}} \mathbb{E}\left[\int_0^\tau h(X(s))ds + \xi(X(\tau))\right] \\
&= \sup_{\tau\in\mathscr{T}} \tilde{\mathbb{E}}\left[\int_0^\tau h(X(s))ds + \xi(X(\tau))\right] \\
&= \sup_{\tau\in\mathscr{T}} \mathbb{E}_0\left[Z(\tau)\left(\int_0^\tau h(X(s))ds + \xi(X(\tau))\right)\right] \\
&= \sup_{\tau\in\mathscr{T}} \mathbb{E}_0\left[\mathbb{E}_0\left[Z(\tau)\,|\mathscr{F}_\tau\right]\left(\int_0^\tau h(X(s))ds + \xi(X(\tau))\right)\right].
\end{aligned} \tag{2.24}$$

From the Bayes point of view, $\mathbb{E}_0\left[Z(t)\,|\mathscr{F}_t\right]$ is the posterior expectation of the Radon-Nikodym derivative $Z(\cdot)$ under the reference probability measure $\mathbb{P}_0$, given the observations of $X(\cdot)$ up-to-date. Because of the independence of ρ, U and $X(\cdot)$ under $\mathbb{P}_0$, from the prior $\mathbb{P}_0$-distributions which are the same as the prior $\mathbb{P}$-distributions (2.4) and (2.5), and by (2.20), the posterior expectation has the form

$$\begin{aligned}
&\mathbb{E}_0\left[Z(t)\,|\mathscr{F}_t\right] \\
&= \sum_{j=1}^m \left(p_j L(t;\mu_j)\int_0^t \frac{L(s;\mu_0)}{L(s;\mu_j)}\lambda e^{-\lambda s}ds\right) + e^{-\lambda t}L(t;\mu_0),\ 0 \le t \le T.
\end{aligned} \tag{2.25}$$

For every $u \in \Theta$, the likelihood ratio process $L(\cdot; u)$ defined in (2.19) is a local $(\mathbb{P}_0, \mathscr{F})$-martingale, because it satisfies the stochastic integral equation

$$L(t;u) = \int_0^t L(s;u)\frac{b(s, X(s); u)}{\sigma^2(s, X(s))}dX(s),\ 0 \le t \le T \tag{2.26}$$

with respect to the local $(\mathbb{P}_0, \mathscr{F})$-martingale $X(\cdot)$. The quadratic variation process of $L(\cdot; u)$ is

$$\langle L(t;u)\rangle = \int_0^t L^2(s;u)\frac{b^2(s,X(s);u)}{\sigma^2(s,X(s))}ds,\ 0\le t\le T. \tag{2.27}$$

On the other hand, from (2.25) we obtain

$$\begin{aligned} &d\left(\mathbb{E}_0\left[Z(t)\,|\mathscr{F}_t\right]\right)\\ &=\sum_{j=1}^m p_j\left(\int_0^t \frac{L(s;\mu_0)}{L(s;\mu_j)}\lambda e^{-\lambda s}ds\right)dL(t;\mu_j)+e^{-\lambda t}dL(t;\mu_0),\ 0\le t\le T, \end{aligned} \tag{2.28}$$

so the posterior expectation $\{\mathbb{E}_0\left[Z(t)\,|\mathscr{F}_t\right]\}_{0\le t\le T}$ is again a local $(\mathbb{P}_0, \mathscr{F})$-martingale.

Lemma 2.1 *For any nonnegative integers n_1, n_2 and n_3, we have*

$$\sup_{\tau\in\mathscr{T}}\mathbb{E}_0\left[L^{n_1}(\tau;\mu_j)R^{n_2}(\tau;\mu_j)\sup_{0\le t\le T}|X(t)|^{n_3}\right]<\infty; \tag{2.29}$$

furthermore, the family

$$\left\{L^{n_1}(\tau;\mu_j)R^{n_2}(\tau;\mu_j)\sup_{0\le t\le T}|X(t)|^{n_3}\right\}_{\tau\in\mathscr{T}} \tag{2.30}$$

is uniformly integrable with respect to the probability measure $\mathbb{P}_0$.

Proof. The proof is purely technical, and is carried out in the Appendix. □

Lemma 2.2 *For $0\le t\le T$, $x\in\mathbb{R}$, $l=(l_0,l_1,\cdots,l_m)\in\mathbb{R}^{m+1}$, and $r=(r_1,\cdots,r_m)\in\mathbb{R}^m$, consider the function*

$$\begin{aligned} \alpha(t,x,l,r) &= \left(\sum_{j=1}^m p_j l_j r_j + e^{-\lambda t}l_0\right)\left(h(x)+\frac{1}{2}\xi''(x)\sigma^2(t,x)\right)\\ &\quad+\left(\sum_{j=1}^m p_j l_j r_j b(t,x;\mu_j)+e^{-\lambda t}l_0 b(t,x;\mu_0)\right)\xi'(x). \end{aligned} \tag{2.31}$$

Then, for $0\le t\le T$, we have

$$\mathbb{E}_0\left[Z(t)\,|\mathscr{F}_t\right]\left(\int_0^t h(X(s))ds+\xi(X(t))\right)=M_0(t)+\int_0^t \alpha\left(s,X(s),L(s),R(s)\right)ds, \tag{2.32}$$

where $M_0(\cdot)$ is some square integrable $(\mathbb{P}_0,\mathscr{F})$-martingale with $M_0(0)=\xi(X(0))$,

$$R(t;\mu_j) := \int_0^t \frac{L(s;\mu_0)}{L(s;\mu_j)}\lambda e^{-\lambda s}ds,\ j=1,\cdots,m, \tag{2.33}$$

$$R(t) := \left(R(t;\mu_1)\cdots,R(t;\mu_m)\right), \tag{2.34}$$

and

$$L(t) := \left(L(t;\mu_0),L(t;\mu_1)\cdots,L(t;\mu_m)\right). \tag{2.35}$$

Proof. Define the process $M_0(\cdot)$ as

$$M_0(t) := \int_0^t \left(\int_0^s h(X(r))dr + \xi(X(s)) \right) d\left(\mathbb{E}_0\left[Z(s)\,|\mathscr{F}_s\right]\right) + \int_0^t \mathbb{E}_0\left[Z(s)\,|\mathscr{F}_s\right] \xi'(X(s))dX(s), \tag{2.36}$$

for $0 \le t \le T$. By (2.28) and Lemma 2.1, $M_0(\cdot)$ is the sum of two integrals of square $\mathbb{P}_0$-integrable processes with respect to local $(\mathbb{P}_0, \mathscr{F})$-martingales, thus also a local $(\mathbb{P}_0, \mathscr{F})$-martingale. By Itô's formula, we have

$$\begin{aligned} & d\left(\mathbb{E}_0\left[Z(t)\,|\mathscr{F}_t\right] \left(\int_0^t h(X(s))ds + \xi(X(t)) \right) \right) \\ =& \left(\int_0^t h(X(s))ds + \xi(X(t)) \right) d\left(\mathbb{E}_0\left[Z(t)\,|\mathscr{F}_t\right]\right) \\ & + \mathbb{E}_0\left[Z(t)\,|\mathscr{F}_t\right] \left(h(X(t)dt + \xi'(X(t))dX(t) + \frac{1}{2}\xi''(X(t))\sigma^2(t, X(t))dt \right) \\ & + \xi''(X(t))d\left\langle \mathbb{E}_0\left[Z(t)\,|\mathscr{F}_t\right], X(t) \right\rangle \\ =& dM_0(t) + \alpha\left(t, X(t), L(t), R(t)\right)dt. \end{aligned} \tag{2.37}$$

We need to show that $M_0(\cdot)$ is a $(\mathbb{P}_0, \mathscr{F})$-martingale, and not just a local martingale. It suffices to show that the family $\{M_0(\tau)\}_{\tau\in\mathscr{T}}$ is uniformly integrable with respect to the probability measure $\mathbb{P}_0$. But $M_0(\cdot)$ can be expressed alternatively as

$$\begin{aligned} M_0(t) =& \left(\sum_{j=1}^m \left(p_j L(t;\mu_j) \int_0^t \frac{L(s;\mu_0)}{L(s;\mu_j)} \lambda e^{-\lambda s} ds \right) + e^{-\lambda t} L(t;\mu_0) \right) \left(\int_0^t h(X(s))ds + \xi(X(t)) \right) \\ & - \int_0^t \alpha\left(s, X(s), L(s), R(s)\right) ds. \end{aligned} \tag{2.38}$$

From the expressions (2.38), (2.31) and Assumption 2.1(3)(5), we know that there exists a constant $C > 0$ and a positive integer n, such that

$$\begin{aligned} & |M_0(t)| \\ \le & C \left(\sum_{j=1}^m L(t;\mu_j)R(t;\mu_j) + L(t;\mu_0) + \int_0^t \sum_{j=1}^m L(s;\mu_j)R(s;\mu_j) + L(s;\mu_0)\, ds \right) \sup_{0\le s\le T} |X(s)|^n, \end{aligned} \tag{2.39}$$

for all $(t, \omega) \in [0, T] \times \Omega$. Then, from Lemma 2.1, we know that, under the probability measure $\mathbb{P}_0$, the local martingale $M_0(\cdot)$ is both square integrable and of class $\mathscr{D}$ on $[0, T]$. The later implies that $M_0(\cdot)$ is a $(\mathbb{P}_0, \mathscr{F})$-martingale.
The process $\alpha\left(\cdot, X(\cdot), L(\cdot), R(\cdot)\right)$ is in fact the instantaneous cross variation process between $\xi(X(\cdot))$ and $\mathbb{E}_0\left[Z(\cdot)\,|\mathscr{F}_\cdot\right]$. □

Lemma 2.3 *For any stopping time $\tau \in \mathscr{T}$,*

$$\begin{aligned} &\mathbb{E}_0 \left[\mathbb{E}_0 \left[Z(\tau) \,|\mathscr{F}_\tau \right] \left(\int_0^\tau h(X(s))ds + \xi(X(\tau)) \right) \right] \\ =&\xi(X(0)) + \mathbb{E}_0 \left[\int_0^\tau \alpha \left(s, X(s), L(s), R(s) \right) ds \right]. \end{aligned} \tag{2.40}$$

Proof. This is because

$$M_0(t) = \mathbb{E}_0 \left[Z(t) \,|\mathscr{F}_t \right] \left(\int_0^t h(X(s))ds + \xi(X(t)) \right) - \int_0^t \alpha \left(s, X(s), L(s), R(s) \right) ds, \quad 0 \le t \le T, \tag{2.41}$$

is a $(\mathbb{P}_0, \mathscr{F})$-martingale, by Lemma 2.2. The optional sampling theorem implies that $\mathbb{E}_0 \left[M_0(\tau) \right] = M_0(0) = \xi(X(0))$, or equivalently, that (2.40) holds. □

Theorem 2.1 *The value of the optimal stopping problem (2.12) can be expressed in terms of the $\mathbb{P}_0$-expectations as*

$$\begin{aligned} V(0) =& \sup_{\tau \in \mathscr{T}} \mathbb{E} \left[\int_0^\tau h(X(s))ds + \xi(X(\tau)) \right] \\ =&\xi(X(0)) + \sup_{\tau \in \mathscr{T}} \mathbb{E}_0 \left[\int_0^\tau \alpha \left(s, X(s), L(s), R(s) \right) ds \right]. \end{aligned} \tag{2.42}$$

Define another value process by

$$V_0(t) := \sup_{\tau \in \mathscr{T}_t} \mathbb{E}_0 \left[\int_t^\tau \alpha \left(s, X(s), L(s), R(s) \right) ds \middle| \mathscr{F}_t \right], \; 0 \le t \le T. \tag{2.43}$$

Then, the optimal $\mathscr{F}$-stopping time τ^* defined in (2.14) has an alternative representation as

$$\tau^* = \inf \left\{ 0 \le t \le T \,| V_0(t) \le 0 \right\}. \tag{2.44}$$

Proof. The first claim can be concluded directly from equality (2.24) and Lemma 2.3. Like (2.14), the expression (2.44) once again comes from the classical results on optimal stopping of continuous processes. □

Lemma 2.4 *(Strong Markov Property) The triple of processes $(X(\cdot), L(\cdot), R(\cdot))$ have the strong Markov property under the measure $\mathbb{P}_0$ with respect to the filtration $\mathscr{F}$.*

Proof. Denoting $\mathbf{1} = (1, 1, \cdots, 1)$ as the $(m+1)$-dimensional row vector of one's, and $\mathbf{0} = (0, \cdots, 0)$ as the m-dimensional row vector of zero's. The triple $(X(\cdot), L(\cdot), R(\cdot))$ constitutes a strong solution to the $(2m+2)$-dimensional SDE

$$\begin{cases} dX(t) =& \sigma(t, X(t)) dW^0(t); \\ dL(t; \mu_j) =& L(t; \mu_j) \dfrac{b(t, X(t); \mu_j)}{\sigma(t, X(t))} dW^0(t), \; j = 0, 1, \cdots, m; \\ dR(t; \mu_j) =& \dfrac{L(t; \mu_0)}{L(t; \mu_j)} \lambda e^{-\lambda t} dt, \; j = 1, \cdots, m \end{cases} \tag{2.45}$$

driven by the standard $\mathbb{P}_0$-Brownian motion $W^0(\cdot)$ of (2.16), with the initial value

$$(X(0), L(0), R(0)) = (x_0, \mathbf{1}, \mathbf{0}). \tag{2.46}$$

From Assumption 2.1 (1)(2)(3), the coefficients of the SDE (2.45) are bounded on compact subsets of $\mathbb{R}^{2m+2}$ and are locally Lipschitz. The SDE (2.45) has a pathwise unique, strong solution. The well-posedness of the SDE (2.45), equivalently the well-posedness of the associated martingale problem, implies the $\mathbb{P}_0$-strong Markov property of $(X(\cdot), L(\cdot), R(\cdot))$, with respect to the Borel sigma algebra $\mathbb{F}$ (Stroock and Varadhan (1997) [18]). But the filtration $\mathscr{F}$ generated by $X(\cdot)$ is contained in $\mathbb{F}$, and the process $(X(\cdot), L(\cdot), R(\cdot))$ is $\mathscr{F}$-adapted. Then the process $(X(\cdot), L(\cdot), R(\cdot))$ has the strong Markov property under the probability measure $\mathbb{P}_0$ with respect to the filtration $\mathscr{F}$. □

The solution $(X(\cdot), L(\cdot), R(\cdot))$ to the SDE (2.45) ranges in $\mathbb{R} \times (0,\infty)^{m+1} \times [0,\infty)^m$.

Lemma 2.5 *(Dynamic Programming Principle)*
There exists a deterministic measurable function $v_0 : [0,T] \times \mathbb{R} \times (0,\infty)^{m+1} \times [0,\infty)^m \to \mathbb{R}$, such that

$$v_0(t, X(t), L(t), R(t)) = \sup_{\tau \in \mathscr{T}_t} \mathbb{E}_0 \left[\int_t^\tau \alpha\left(s, X(s), L(s), R(s)\right) ds \,\middle|\, \mathscr{F}_t \right], \ 0 \le t \le T. \tag{2.47}$$

For any stopping time $\tau_0 \in \mathscr{T}_t$, the dynamic programming principle

$$\begin{aligned} &v_0(t, X(t), L(t), R(t)) \\ &= \sup_{\tau \in \mathscr{T}_t} \mathbb{E}_0 \left[\int_t^{\tau \wedge \tau_0} \alpha\left(s, X(s), L(s), R(s)\right) ds + v_0\left(\tau_0, X(\tau_0), L(\tau_0), R(\tau_0)\right) \mathbb{1}_{\{\tau_0 \le \tau\}} \,\middle|\, \mathscr{F}_t \right] \end{aligned} \tag{2.48}$$

holds for $0 \le t \le T$.

Theorem 2.2 *For any function $\psi : [0,T] \times \mathbb{R} \times \mathbb{R}^{m+1} \times \mathbb{R}^m \to \mathbb{R}$, $(t,x,l,r) \mapsto \psi(t,x,l,r)$, denote the infinitesimal generator*

$$\begin{aligned} &\mathscr{A}\psi(t,x,l,r) \\ :=& \left(\sum_{j=1}^m \frac{l_0}{l_j} \lambda e^{-\lambda t} \frac{\partial}{\partial r_j} + \frac{1}{2}\sigma^2(t,x) \frac{\partial^2}{\partial x^2} + \frac{1}{2} \sum_{j=0}^m l_j b(t,x;\mu_j) \left(\frac{\partial^2}{\partial x \partial l_j} + \frac{\partial^2}{\partial l_j \partial x} \right) \right. \\ &\left. + \frac{1}{2} \sum_{j,k=0}^m l_j l_k \frac{b(t,x;\mu_j) b(t,x;\mu_k)}{\sigma^2(t,x)} \frac{\partial^2}{\partial l_j \partial l_k} \right) \psi(t,x,l,r). \end{aligned} \tag{2.49}$$

The value function v_0 defined in (2.47) is a viscosity solution (Definition 4.1 (1)) to the variational inequality

$$\min \left\{ -\left(\frac{\partial}{\partial t} v_0 + \mathscr{A} v_0 + \alpha \right)(t,x,l,r), \left(v_0 + \alpha\right)(t,x,l,r) \right\} = 0, \tag{2.50}$$

with the terminal condition

$$v_0(T,x,l,r) = 0, \tag{2.51}$$

for all $0 \le t \le T$, $(x,l,r) \in \mathbb{R} \times (0,\infty)^{m+1} \times [0,\infty)^m$.

Proof. The viscosity solution property is derived from the dynamic programming principle Lemma 2.5. The readers may refer to section V of [6] Fleming and Soner (1993) for the derivation of second order HJB PDEs, and to section 5 of [14] Pham (2009) for optimal stopping problems. □

Remark. The coefficient of the second-order terms of the variational inequality (2.50) does not satisfy the uniform ellipticity condition. Existence, uniqueness and regularity properties of classical solutions to degenerate PDEs and variational inequalities have been an entire rich open area of research. For the purpose of solving the optimization and control problems in this paper, we shall content ourselves with solutions in the viscosity sense.

2.3 Computing the Value Function

If we denote

$$y = (x, l_0, l_1, \cdots, l_m, r_1, \cdots, r_m), \tag{2.52}$$

$$b_Y(t, y) = \left(0, 0, 0, \cdots, 0, \frac{l_0}{l_1}\lambda e^{-\lambda t}, \cdots, \frac{l_0}{l_m}\lambda e^{-\lambda t}\right), \tag{2.53}$$

and

$$\sigma_Y(t, y) = \left(\sigma(t, x), l_0\frac{b(t, x; \mu_0)}{\sigma(t, x)}, l_1\frac{b(t, x; \mu_1)}{\sigma(t, x)}, \cdots, l_m\frac{b(t, x; \mu_m)}{\sigma(t, x)}, 0, \cdots, 0\right), \tag{2.54}$$

for all $(t, y) = (t, x, l, r)$ in $[0, T] \times \mathbb{R} \times \mathbb{R}^{m+1} \times \mathbb{R}^m$, then the SDE (2.45), which has a pathwise unique, strong solution

$$Y(\cdot) = (X(\cdot), L(\cdot; \mu_0), L(\cdot; \mu_1), \cdots, L(\cdot; \mu_m), R(\cdot; \mu_1), \cdots, R(\cdot; \mu_m)), \tag{2.55}$$

can be written in the vector form

$$dY(t) = b_Y(t, Y(t))dt + \sigma_Y(t, Y(t))dW^0(t), \ 0 \le t \le T \tag{2.56}$$

with the initial value

$$Y(0) = (x_0, \mathbf{1}, \mathbf{0}), \tag{2.57}$$

where $W^0(\cdot)$ is the standard $\mathbb{P}_0$-Brownian motion of (2.16). The generator (2.49) of the second-order parabolic variational inequality (2.50) can be written as

$$\mathscr{A}\psi(t, y) = \left(b_Y \cdot D\psi + \frac{1}{2}\text{ trace }\left(\sigma_Y\sigma_Y^{\text{tr}}D^2\psi\right)\right)(t, y), \tag{2.58}$$

where σ_Y^{tr} denotes the matrix transpose of the row vector σ_Y. The coefficients b_Y and σ_Y of the variational inequality (2.50) is locally Lipschitz, but not necessarily globally Lipschitz, in the space variable over the unbounded domain $\mathbb{R} \times (0, \infty)^{m+1} \times [0, \infty)^m$, which is the range of $Y(\cdot) = (X(\cdot), L(\cdot), R(\cdot))$. Uniqueness of viscosity solutions to such variational inequalities remains open, hence the numerical implementation of the variational inequality (2.50) is a question. In order to enable numerical computation of the value function v_0 as in (2.47), we shall approximate it with a sequence $\{v_n\}_{n=1}^{\infty}$ of functions which are unique viscosity solutions to some variational inequalities. The sequence of variational inequalities will be derived from optimal stopping problems of the type (2.12) with finite exit boundaries for the augmented state processes.

For every $n = 1, 2, \cdots$, define the bounded open domain

$$\mathscr{O}_n = (-n, n) \times \left(\frac{1}{n}, n\right)^{m+1} \times \left(-\frac{1}{n}, n\right)^m \subseteq \mathbb{R}^{2m+2}, \tag{2.59}$$

and the exit time

$$T_n = \inf\{0 \le t \le T | Y(t) \notin \mathscr{O}_n\} \wedge T. \tag{2.60}$$

Denote by $\mathscr{T}^n$ the set of stopping times with values between 0 and T_n. Instead of (2.12), the optimization problem that we shall consider is looking for an $\mathscr{T}$-stopping time τ_n^* to achieve the supremum in

$$\sup_{\tau \in \mathscr{T}^n} \mathbb{E}\left[\int_0^\tau h(X(s))ds + \xi(X(\tau))\right] = \sup_{\tau \in \mathscr{T}} \mathbb{E}\left[\int_0^{\tau \wedge T_n} h(X(s))ds + \xi(X(\tau))\right]. \tag{2.61}$$

As $n \to \infty$, the open domain $\mathscr{O}_n$ converges to $\mathbb{R} \times (0, \infty)^{m+1} \times [0, \infty)^m$, hence

$$\lim_{n \to \infty} \uparrow T_n = T. \tag{2.62}$$

The following two theorems suggest approximating the value function v_0 as in (2.47) by the unique viscosity solutions v_n to a sequence of variational inequalities.

Theorem 2.3 *There exists a function $v_n : [0, T] \times \bar{\mathscr{O}}_n \to \mathbb{R}$, such that*

$$v_n(t, Y(t)) = \sup_{\tau \in \mathscr{T}_t} \mathbb{E}_0 \left[\int_t^{\tau \wedge T_n} \alpha(s, Y(s))\, ds \middle| \mathscr{F}_t\right],\ 0 \le t \le T. \tag{2.63}$$

The value function v_n is a viscosity solution to the variational inequality

$$\min\left\{-\left(\frac{\partial}{\partial t} v_n + \mathscr{A} v_n + \alpha\right)(t, y), (v_n + \alpha)(t, y)\right\} = 0,\ (t, y) \in Q_n \setminus \partial^* Q_n, \tag{2.64}$$

with boundary condition

$$v_n(t, y) = 0,\ (t, y) \in \partial^* Q_n, \tag{2.65}$$

where

$$Q_n := (0, T) \times \mathscr{O}_n \tag{2.66}$$

and

$$\partial^* Q_n := (\{T\} \times \mathscr{O}_n) \cup ([0, T] \times \partial \mathscr{O}_n). \tag{2.67}$$

As $n \to \infty$, the value function v_n converges pointwise to the value function v_0 defined as in (2.47).

Proof. Following the same kind of arguments in subsection 2.2 that lead to Lemma 2.5 and Theorem 2.2, we may conclude (2.63) and the viscosity solution property of v_n as solution to the variational inequality (2.64). From the polynomial growth rate of the function α defined in (2.31) and Lemma 2.1, the dominated convergence theorem guarantees the convergence of v_n to v_0. $\square$

Lemma 2.6 *The value function v_n as in (2.63) is Lipschitz over the bounded domain $\bar{\mathscr{O}}_n$, uniformly for all $0 \le t \le T$, meaning that there exists a constant $C_n > 0$, such that*

$$|v_n(t, y^1) - v_n(t, y^2)| \le C_n |y^1 - y^2|, \text{ for all } (t, y^1), (t, y^2) \in [0, T] \times \bar{\mathscr{O}}_n. \tag{2.68}$$

Proof. Because of the local Lipschitz continuity, Assumption 2.6(1) of the coefficients b and σ, and Assumption 2.1(5) of the rewards h, ξ' and ξ'', the function α defined in (2.31) is also locally Lipschitz. Let $Y^y(t)$ denote the value of the solution to the SDE (2.56) at time t with initial value $Y(0) = y \in \bar{\mathscr{O}}_n$. From the estimate

$$\mathbb{E}_0 \left[\left| Y^{y^1}(t) - Y^{y^2}(t) \right| \right] \le C|y^1 - y^2|, \tag{2.69}$$

it follows that v_n is uniformly Lipschitz. (c.f. section 3 of [13] Pham (1998)). $\square$

The collection of all uniformly Lipschitz functions over $[0, T] \times \bar{\mathscr{O}}_n$ is denoted as $\mathscr{C}_{\text{Lip}}([0, T] \times \bar{\mathscr{O}}_n)$.

Theorem 2.4 *Suppose $\underline{v}_n$ and $\bar{v}_n$ in $\mathscr{C}_{Lip}([0, T] \times \bar{\mathscr{O}}_n)$ are, respectively, a viscosity subsolution and a viscosity supersolution to the variational inequality (2.64) with the boundary condition (2.65), then the comparison result*

$$\sup_{\bar{Q}_n} (\underline{v}_n - \bar{v}_n) = \sup_{\partial^* Q_n} (\underline{v}_n - \bar{v}_n) = 0 \tag{2.70}$$

holds, hence the viscosity solution v_n to (2.64), (2.65) is unique.

Proof. (outline) For variational inequalities with Lipschitz coefficients, the proof for uniqueness of viscosity solutions is a streamlined procedure using the Crandall-Ishii maximum principle. Our proof is tailored on Lemma 7.1 and Theorem 8.1 for second-order parabolic HJB PDEs with Lipschitz coefficients, in section V of [6] Fleming and Soner (1993), in the absence of the controls. To adjust for the variational inequality, when subtracting the two inequalities which define the subsolution and the supersolution, we shall use the fact that $\min\{a, b\} - \min\{c, d\} \le 0$ implies that either $a \le c$ or $b \le d$, as in the proof of Theorem 4.1 in [13] Pham (1998).
For completeness of the exposition, a detailed proof is provided in the Appendix. $\square$

The solution to the original variational inequality of (2.50) may or may not be unique. The one of interest is the solution v_0 which is the limit

$$v_0 = \lim_{n \to \infty} v_n. \tag{2.71}$$

Theorem 2.3 and Theorem 2.4 suggest that the value function v_0 as in (2.47) can be approximated as the limit (2.71), and that each v_n is the unique solution to the numerically implementable variational inequality (2.64). Having solved the variational inequalities, the optimal stopping time and the value function of the optimization problem (2.12) are then obtained in terms of the limit v_0 of the solutions v_n, and of the triple $(X(\cdot), L(\cdot), R(\cdot))$. The triple $(X(\cdot), L(\cdot), R(\cdot))$ of processes, which is adapted to the filtration $\mathscr{F}$ generated by the observation $X(\cdot)$, can be viewed as a "sufficient statistic" for the optimization problem (2.12). This "sufficient statistic" that the decision maker need to monitor remains the same for all cumulative reward functions $h(\cdot)$ and all terminal reward functions $\xi(\cdot)$ in (2.12).

Proposition 2.1 *The value function of the optimal stopping problem (2.12) can be calculated as*

$$\sup_{\tau \in \mathscr{T}_t} \mathbb{E}\left[\int_t^\tau h(X(s))ds + \xi(X(\tau)) \middle| \mathscr{F}_t \right] = \xi(X(t)) + v_0(t, X(t), L(t), R(t)), \tag{2.72}$$

for all $0 \le t \le T$. *In particular,*

$$\sup_{\tau \in \mathscr{T}} \mathbb{E}\left[\int_0^\tau h(X(t))dt + \xi(X(\tau)) \right] = \xi(X(0)) + v_0(0, X(0), \mathbf{1}, \mathbf{0}). \tag{2.73}$$

The optimal $\mathscr{F}$*-stopping time* τ^* *in (2.14) that achieves the supremum in (2.12) has another representation as*

$$\tau^* = \inf \left\{ 0 \le t \le T \,|v_0(t, X(t), L(t), R(t)) \le 0 \right\}. \tag{2.74}$$

Proof. Use Theorem 2.1 and Lemma 2.5. □

2.4 Examples

Example 2.3 *(Brownian motion with drift uncertainty, continued)*
Let the diffusion $X(\cdot)$ *be the drifted Brownian motion described in Example 2.1. We look for an* $\mathscr{F}$*-stopping time* τ^* *to achieve the supremum*

$$\sup_{\tau \in \mathscr{T}} \mathbb{E}\left[X(\tau) \right]. \tag{2.75}$$

Solution. The likelihood ratio process in (2.19) has the form

$$L(t; u) = \exp\left\{ uX(t) - \frac{1}{2}u^2 t \right\}, \; 0 \le t \le T, \tag{2.76}$$

for $u = \mu_0, \mu_+, \mu_-$. Under the $\mathbb{P}_0$-measure defined in (2.21) via the Radon-Nikodym derivative

$$\begin{aligned} Z(t) &= \exp\left\{ \int_0^t \theta(s)dX(s) - \frac{1}{2}\int_0^t \theta^2(s)ds \right\} \\ &= \left(\mathbb{1}_{\{U=\mu_+\}} L(t;\mu_+) \frac{L(\rho;\mu_0)}{L(\rho;\mu_+)} + \mathbb{1}_{\{U=\mu_-\}} L(t;\mu_-) \frac{L(\rho;\mu_0)}{L(\rho;\mu_-)} \right) \mathbb{1}_{\{\rho<t\}} + \mathbb{1}_{\{\rho \ge t\}} L(t;\mu_0), \end{aligned} \tag{2.77}$$

the process $X(\cdot)$ is a standard Brownian motion, and the likelihood ratio process $L(\cdot)$ is a geometric Brownian motion. Calculating the posterior expectation of $Z(\cdot)$ as in (2.25),

$$\begin{aligned}&\mathbb{E}_0\left[Z(t)\,|\mathscr{F}_t\right]\\&=\left(p_+L(t;\mu_+)\int_0^t\frac{L(s;\mu_0)}{L(s;\mu_+)}\lambda e^{-\lambda s}ds+p_-L(t;\mu_-)\int_0^t\frac{L(s;\mu_0)}{L(s;\mu_-)}\lambda e^{-\lambda s}ds\right)+e^{-\lambda t}L(t;\mu_0).\end{aligned}\tag{2.78}$$

From (2.31),

$$\alpha(t,x,l,r)=p_+\mu_+l_+r_++p_-\mu_-l_-r_-+\mu_0e^{-\lambda t}l_0.\tag{2.79}$$

Define

$$R(t;\mu_\pm):=\int_0^t\frac{L(s;\mu_0)}{L(s;\mu_\pm)}\lambda e^{-\lambda s}ds,\ 0\le t\le T.\tag{2.80}$$

The value function

$$\begin{aligned}&v_0(t,X(t),L(t),R(t))\\&=\sup_{\tau\in\mathscr{T}_t}\mathbb{E}_0\left[\int_t^\tau\left(p_+\mu_+L(s;\mu_+)R(s;\mu_+)+p_-\mu_-L(s;\mu_-)R(s;\mu_-)+\mu_0e^{-\lambda t}L(s;\mu_0)\right)ds\middle|\mathscr{F}_t\right]\end{aligned}\tag{2.81}$$

is a viscosity solution to the variational inequality (2.50) with infinitesimal generator

$$\begin{aligned}&\mathscr{A}v_0(t,x,l,r)\\&=\left(\lambda e^{-\lambda t}\left(\frac{l_0}{l_+}\frac{\partial}{\partial r_+}+\frac{l_0}{l_-}\frac{\partial}{\partial r_-}\right)+\frac{1}{2}\frac{\partial^2}{\partial x^2}\right.\\&\quad+\frac{1}{2}\left(\mu_+^2l_+^2\frac{\partial^2}{\partial l_+^2}+\mu_+\mu_-l_+l_-\left(\frac{\partial^2}{\partial l_+\partial l_-}+\frac{\partial^2}{\partial l_-\partial l_+}\right)+\mu_-^2l_-^2\frac{\partial^2}{\partial l_-^2}\right)\\&\quad\left.+\frac{1}{2}\mu_+l_+\left(\frac{\partial^2}{\partial x\partial l_+}+\frac{\partial^2}{\partial l_+\partial x}\right)+\frac{1}{2}\mu_-l_-\left(\frac{\partial^2}{\partial x\partial l_-}+\frac{\partial^2}{\partial l_-\partial x}\right)\right)v_0(t,x,l,r).\end{aligned}\tag{2.82}$$

Expression (2.76) of the likelihood ratio suggests reduction of dimensionality for the variational inequality (2.50). For $(t,x,r)\in[0,T]\times\mathbb{R}\times[0,\infty)^2$, define the function

$$\begin{aligned}&\bar{\alpha}(t,x,r)\\=&p_+\mu_+\exp\left\{\mu_+x-\frac{1}{2}\mu_+^2t\right\}r_++p_-\mu_-\exp\left\{\mu_-x-\frac{1}{2}\mu_-^2t\right\}r_-+\mu_0e^{-\lambda t}\exp\left\{\mu_0x-\frac{1}{2}\mu_0^2t\right\}.\end{aligned}\tag{2.83}$$

By (2.76), the value function (2.81) becomes

$$v_0(t,X(t),L(t),R(t))=\bar{v}_0(t,X(t),R(t))=\sup_{\tau\in\mathscr{T}_t}\mathbb{E}_0\left[\int_t^\tau\bar{\alpha}(s,X(s),R(s))ds\middle|\mathscr{F}_t\right],\tag{2.84}$$

a deterministic function $\bar{v}_0$ of the time t, the state processes $X(t)$ and the augmented variables $R(t;\mu_+)$ and $R(t;\mu_-)$ only. The likelihood ratios L have disappeared. The function $\bar{v}_0$ is a viscosity solution to the variational inequality

$$\begin{cases} \min\left\{ -\left(\frac{\partial}{\partial t}\bar{v}_0 + \bar{\mathscr{A}}\bar{v}_0 + \bar{\alpha}\right)(t,x,r), (\bar{v}_0 + \bar{\alpha})(t,x,r) \right\} = 0; \\ v_0(T,x,r) = 0, \end{cases} \tag{2.85}$$

for $0 \le t \le T$, $(x,r) \in \mathbb{R} \times [0,\infty)^2$, where the infinitesimal generator $\bar{\mathscr{A}}$ has the form

$$\bar{\mathscr{A}}\bar{v}_0(t,x,r) = \left(\frac{1}{2}\frac{\partial^2}{\partial x^2} + \lambda e^{-\lambda t}\left(\exp\left\{ (\mu_0 - \mu_+)x - \frac{1}{2}(\mu_0^2 - \mu_+^2)t \right\} \frac{\partial}{\partial r_+} \right.\right.$$
$$\left.\left. + \exp\left\{ (\mu_0 - \mu_-)x - \frac{1}{2}(\mu_0^2 - \mu_+^2)t \right\} \frac{\partial}{\partial r_-} \right)\right) \bar{v}_0(t,x,r). \tag{2.86}$$

□

Example 2.4 *(Geometric Brownian motion with drift uncertainty, continued)*
The diffusion $X(\cdot)$ is the geometric Brownian motion in Example 2.2. The goal is again to find an $\mathscr{F}$-stopping time τ^* that achieves the supremum in (2.75).

Solution. We may compute to get the likelihood ratio processes

$$L(t;u) = \exp\left\{ \frac{u}{\sigma^2}\int_0^t \frac{dX(s)}{X(s)} - \frac{u^2}{2\sigma^2}t \right\},\ 0 \le t \le T, \tag{2.87}$$

for $u \in \Theta$, and the Radon-Nikodym derivative

$$\begin{aligned} Z(t) &= \exp\left\{ \int_0^t \frac{\theta(s)}{\sigma^2 X(s)} dX(s) - \frac{1}{2}\int_0^t \frac{\theta(s)}{\sigma^2} ds \right\} \\ &= \left(\sum_{j=1}^m \mathbb{1}_{\{U=\mu_j\}} L(t;\mu_j)\frac{L(\rho;\mu_0)}{L(\rho;\mu_j)} \right) \mathbb{1}_{\{\rho<t\}} + L(t;\mu_0)\mathbb{1}_{\{\rho \ge t\}},\ 0 \le t \le T. \end{aligned} \tag{2.88}$$

The posterior expectation $\mathbb{E}_0\left[Z(t)\,|\mathscr{F}_t\right]$ has the same expression as (2.78), the augmented state variable $R(t)$ is defined as in (2.33) and (2.34), the function α in (2.31) becomes

$$\alpha(t,x,l,r) = x\left(\sum_{j=1}^m p_j \mu_j l_j r_j \right) + \mu_0 e^{-\lambda t} x l_0, \tag{2.89}$$

and the value function

$$\begin{aligned} &v_0(t, X(t), L(t), R(t)) \\ &= \sup_{\tau \in \mathscr{T}_t} \mathbb{E}_0\left[\int_t^\tau X(s)\left(\sum_{j=1}^m p_i \mu_j L(s;\mu_j) R(s;\mu_j) + \mu_0 e^{-\lambda t} L(s;\mu_0) ds \right) \middle| \mathscr{F}_t \right] \end{aligned} \tag{2.90}$$

is a viscosity solution to the variational inequality (2.50) with infinitesimal generator

$$\begin{aligned}\mathscr{A} v_0(t,x,l,r) = \Bigg(& \lambda e^{-\lambda t} \sum_{j=1}^{m} \frac{l_0}{l_j} \frac{\partial}{\partial r_j} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + \frac{1}{2} \sum_{j,k=0}^{m} \frac{\mu_j \mu_k}{\sigma^2} l_j l_k \frac{\partial^2}{\partial l_j \partial l_k} \\ & + \frac{1}{2} \sum_{j=0}^{m} \mu_j l_j x \left(\frac{\partial^2}{\partial x \partial l_j} + \frac{\partial^2}{\partial l_j \partial x} \right) \Bigg) v_0(t,x,l,r),\end{aligned} \tag{2.91}$$

for $0 \le t \le T$, $(x,l,r) \in (0,\infty) \times (0,\infty)^{m+1} \times [0,\infty)^m$.
The solution in this geometric Brownian motion case connects with the routine practice in the filtering theory via the change of variable

$$\bar{X}(t) = \int_0^t \frac{dX(s)}{X(s)} = \int_0^t \theta(s) ds + \sigma W(t),\ 0 \le t \le T. \tag{2.92}$$

The process $\bar{X}(\cdot)$ is a $(\mathbb{P}_0, \mathscr{F})$-martingale with quadratic variation $\sigma^2 t$. From the change of variable (2.92), the likelihood ratios in (2.87) can be rewritten as

$$L(t;u) = \exp\left\{ \frac{u}{\sigma^2} \bar{X}(t) - \frac{u^2}{2\sigma^2} t \right\},\ 0 \le t \le T, \tag{2.93}$$

for $u \in \Theta$. It follows that

$$R(t;\mu_j) = \int_0^t \lambda e^{-\lambda s} \exp\left\{ \frac{\mu_0 - \mu_j}{\sigma^2} \bar{X}(s) - \frac{\mu_0^2 - \mu_j^2}{2\sigma^2} s \right\} ds,\ 0 \le t \le T, \tag{2.94}$$

for $j = 1, \cdots, m$. For $(t,x,\bar{x},r) \in [0,T] \times (0,\infty) \times \mathbb{R} \times [0,\infty)^m$, define the function

$$\bar{\alpha}(t,x,\bar{x},r) = x \sum_{j=1}^{m} p_j \mu_j \exp\left\{ \frac{\mu_j}{\sigma^2} \bar{x} - \frac{\mu_j^2}{2\sigma^2} t \right\} r_j + \mu_0 e^{-\lambda t} \exp\left\{ \frac{\mu_0}{\sigma^2} \bar{x} - \frac{\mu_0^2}{2\sigma^2} t \right\}. \tag{2.95}$$

By (2.93) and (2.94), the value function (2.90) becomes

$$v_0(t, X(t), L(t), R(t)) = \bar{v}_0(t, X(t), \bar{X}(t), R(t)) = \sup_{\tau \in \mathscr{T}_t} \mathbb{E}_0 \left[\int_t^\tau \bar{\alpha}(s, X(s), \bar{X}(t), R(s)) ds \middle| \mathscr{F}_t \right], \tag{2.96}$$

a deterministic function $\bar{v}_0$ of time t, state processes $X(t)$ and the augmented processes $\bar{X}(t)$ and $R(t)$. The $(m+1)$-dimensional likelihood ratio process $L(\cdot)$ as argument of v_0 is replaced by the one-dimensional process $\bar{X}(\cdot)$ as argument of $\bar{v}_0$. This reduces m dimensions. The function $\bar{v}_0$ is a viscosity solution to the variational inequality

$$\begin{cases} \min\left\{ -\left(\frac{\partial}{\partial t} \bar{v}_0 + \bar{\mathscr{A}} \bar{v}_0 + \bar{\alpha} \right)(t,x,\bar{x},r), \left(\bar{v}_0 + \bar{\alpha} \right)(t,x,\bar{x},r) \right\} = 0; \\ \bar{v}_0(T,x,\bar{x},r) = 0, \end{cases} \tag{2.97}$$

for $0 \le t \le T$, $(x, \bar{x}, r) \in (0,\infty) \times \mathbb{R} \times [0,\infty)^m$, where the infinitesimal generator $\bar{\mathscr{A}}$ has the form

$$\begin{aligned} &\bar{\mathscr{A}}\bar{v}_0(t,x,\bar{x},r) \\ =&\Bigg(\frac{1}{2}\sigma^2x^2\frac{\partial^2}{\partial x^2} + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial \bar{x}^2} + \frac{1}{2}\sigma^2 x\left(\frac{\partial^2}{\partial x \partial \bar{x}} + \frac{\partial^2}{\partial \bar{x}\partial x}\right) \\ &+ \lambda e^{-\lambda t}\exp\left\{\frac{\mu_0-\mu_j}{\sigma^2}\bar{x} - \frac{1}{2}\frac{\mu_0^2-\mu_j^2}{\sigma^2}t\right\}\frac{\partial}{\partial r_j}\Bigg)\bar{v}_0(t,x,\bar{x},r). \end{aligned} \tag{2.98}$$

The dimensionality of the variational inequality can be further reduced, by using an alternative expression

$$L(t;\mu_j) = \exp\left\{\frac{1}{2}\left(\mu_j - \frac{\mu_j^2}{\sigma^2}\right)t\right\}X(t)^{\frac{\mu_j}{\sigma^2}},\ 0 \le t \le T, \tag{2.99}$$

of the likelihood ratio processes (2.99), in terms of $X(t)$. Then,

$$R(t;\mu_j) = \int_0^t \lambda e^{-\lambda s}\sum_{j=1}^m \exp\left\{\frac{1}{2}\left(\mu_0-\mu_j-\frac{\mu_0^2-\mu_j^2}{\sigma^2}\right)s\right\}X(s)^{\frac{\mu_0-\mu_j}{\sigma^2}}ds,\ 0\le t\le T, \tag{2.100}$$

for $j = 1, \cdots, m$. Define the function

$$\bar{\bar{\alpha}}(t,x,y,r) = \sum_{j=1}^m p_j\mu_j x^{\frac{\mu_j}{\sigma^2}+1}r_j + \mu_0 e^{-\lambda t}x^{\frac{\mu_0}{\sigma^2}+1}. \tag{2.101}$$

By (2.99) and (2.100), the value function (2.90) can be written as

$$v_0(t,X(t),L(t),R(t)) = \bar{\bar{v}}_0(t,X(t),R(t)) = \sup_{\tau\in\mathscr{T}_t}\mathbb{E}_0\left[\int_t^\tau \bar{\bar{\alpha}}(s,X(s),R(s))ds\,\middle|\,\mathscr{F}_t\right]. \tag{2.102}$$

The deterministic function $\bar{\bar{v}}_0$: $[0,T]\times(0,\infty)\times[0,\infty)^m \to \mathbb{R}$, $(t,x,r)\mapsto \bar{\bar{v}}_0(t,x,r)$, is a viscosity solution to the variational inequality

$$\begin{cases} \min\left\{-\left(\frac{\partial}{\partial t}\bar{\bar{v}}_0 + \bar{\bar{\mathscr{A}}}\bar{\bar{v}}_0 + \bar{\bar{\alpha}}\right)(t,x,r), \left(\bar{\bar{v}}_0 + \bar{\bar{\alpha}}\right)(t,x,r)\right\} = 0; \\ \bar{\bar{v}}_0(T,x,r) = 0, \end{cases} \tag{2.103}$$

for $0 \le t \le T$, $(x,r)\in(0,\infty)\times[0,\infty)^m$, with the infinitesimal generator

$$\begin{aligned} &\bar{\bar{\mathscr{A}}}\bar{\bar{v}}_0(t,x,r) \\ =&\left(\frac{1}{2}\sigma^2x^2\frac{\partial^2}{\partial x^2} + \lambda e^{-\lambda t}\sum_{j=1}^m\exp\left\{\frac{1}{2}\left(\mu_0-\mu_j-\frac{\mu_0^2-\mu_j^2}{\sigma^2}\right)t\right\}x^{\frac{\mu_0-\mu_j}{\sigma^2}}\frac{\partial}{\partial r_j}\right)\bar{\bar{v}}_0(t,x,r). \end{aligned} \tag{2.104}$$

Compared to the arguments of v_0, the $(m+1)$-dimensional likelihood ratio variable has disappeared from $\bar{\bar{v}}_0$. □

2.5 Comparison with Robust Optimization

The Bayes sequential approach that we take incorporates implicit parameter estimation into decision making, based on the *a priori* knowledge about the parameter. Comparing the variational inequality (2.50) with the one for the counterpart optimal stopping problem without model uncertainty, the additional variables l and r in the arguments of the value function v_0, together with the additional terms in the generator (2.49), are extra efforts required to estimate the jump time ρ and the new value U of the parameter $\theta(\cdot)$.

To cope with model uncertainty, an alternative philosophy to the Bayes sequential approach is the robust optimization, which conducts maximization in the worst-case scenarios. A robust optimal stopping time is one of the Brownian filtration $\mathscr{F}^W$, a stopping time that achieves the supremum in

$$\sup_{\tau \in\ \mathscr{F}^W\text{-stopping times}} \inf_{\rho,\ u} \mathbb{E}\left[\int_0^\tau h(X(t))dt + \xi(X(\tau))\right], \tag{2.105}$$

instead of in (2.12). The infimum in (2.105) is taken over all distributions with nonnegative support for ρ, and all the possible values $u \in \Theta$. As a variant of discretionary stopping, the problem (2.105) can be viewed as a zero-sum game between a stopper who chooses the stoping time τ and a controller who decides the regime switch by choosing ρ and u, the type of zero-sum game of control and stopping studied in [10] by Karatzas and Zamfirescu (2008).

Particularly in the context of Examples 2.3 and 2.4, the goal (2.105) becomes

$$\sup_{\tau \in\ \mathscr{F}^W\text{-stopping times}} \inf_{\rho,\ u} \mathbb{E}\left[X(\tau)\right]. \tag{2.106}$$

The parameter space Θ is $\{\mu_0, \mu_+, \mu_-\}$ in Example 2.3, and is $\{\mu_0, \mu_1, \cdots, \mu_m\}$ in Example 2.4.

In Example 2.3, the infimum in (2.106) is achieved by

$$\mu^*_{\text{robust}} = \min\{\mu_0, \mu_+, \mu_-\}, \tag{2.107}$$

and

$$\rho^*_{\text{robust}} = \begin{cases} 0, & \mu_0 > \mu^*_{\text{robust}}; \\ T, & \mu_0 \leq \mu^*_{\text{robust}}. \end{cases} \tag{2.108}$$

The goal (2.106) then becomes finding a stopping time τ^*_{robust} to achieve the supremum

$$\sup_{\tau \in\ \mathscr{F}^W\text{-stopping times}} \mathbb{E}\left[\mu^*_{\text{robust}}\tau + W(\tau)\right]. \tag{2.109}$$

In Example 2.4, the infimum in (2.106) is achieved by

$$\mu^*_{\text{robust}} = \min\{\mu_0, \mu_1, \cdots, \mu_m\}, \tag{2.110}$$

and ρ^*_{robust} as in (2.108). We need to find a stopping time τ^*_{robust} to achieve the supremum

$$\sup_{\tau \in \mathscr{F}^W\text{-stopping times}} \mathbb{E}\left[\exp\left\{\left(\mu^*_{\text{robust}} - \frac{1}{2}\sigma^2\right)\tau + W(\tau)\right\}\right]. \tag{2.111}$$

Both (2.109) and (2.111) are optimal stopping problems of supermartingales or submartingales of the Brownian motion $W(\cdot)$, depending on whether $\mu^*_{\text{robust}} \le 0$ or $\mu^*_{\text{robust}} \ge 0$. The resulted robust optimal stopping time is the same for the two examples, being

$$\tau^*_{\text{robust}} = \begin{cases} 0, \ \mu^*_{\text{robust}} < 0; \\ \text{any time on } [0,T], \mu^*_{\text{robust}} = 0; \\ T, \ \mu^*_{\text{robust}} > 0. \end{cases} \tag{2.112}$$

In both examples, if $\mu^*_{\text{robust}} > 0$, each realization of the parameter $\theta(\cdot)$ produces a submartingale, then the optimal stopping problem is trivial. In the case $\mu^*_{\text{robust}} < 0$, if at least one number in the parameter space Θ is positive, there is still chance to stop at some time for an average reward greater than $X(0) = x_0$. Since $\tau^*_{\text{robust}} = 0$ is also a stopping time in $\mathscr{T}$, the Bayes sequential optimal stopping rule τ^* will stop for an expected reward no lower than x_0. The optimal value from the Bayes sequential approach can be computed buy solving the variational inequalities (2.50) with the infinitesimal generator (2.91) and the terminal condition (2.51), following the approximation scheme suggested in section 2.3. Speaking of the Brownian motion and the geometric Brownian motion examples with possible negative drift, robust optimization makes a most conservative decision:" If you consider all the terrible things that can happen to you during the day, what is the point of getting out in the morning? "

3 Impulse Control

3.1 The Model

With $\tau_0 = 0$ and for every $i = 1, 2, \cdots, N$, inductively specify a stopping time τ_i with respect to the filtration $\mathscr{F} = \{\mathscr{F}_t\}_{0 \le t \le T}$, which is generated by the process $X(\cdot)$. Let $X(0) = x_0$. On the time interval $t \in (\tau_{i-1}, \tau_i)$, the process $X(\cdot)$ evolves according to the drift-uncertain diffusion (2.1) described in subsection 2.1. The coefficients b and σ satisfy Assumption 2.1 (1)(2)(3). In the differential form, the process $X(\cdot)$ is the pathwise unique, strong solution to the stochastic differential equation

$$dX(t) = b(t, X(t); \theta(t))dt + \sigma(t, X(t))dW(t), \ \tau_{i-1} < t < \tau_i, \tag{3.1}$$

with the initial value $X(\tau_{i-1})$. At every stopping time τ_i, there is an intervention ζ_i, that is, an $\mathbb{R}$-valued $\mathscr{F}_{\tau_i-}$ - measurable random variable which causes the process $X(\cdot)$ to jump at a predictable size

$$X(\tau_i) - X(\tau_i-) = \gamma(X(\tau_i-), \zeta_i) < \infty. \tag{3.2}$$

The jump size $\gamma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a deterministic measurable function whose growth rate will be specified in next subsection. The pair (τ_i, ζ_i) consisting of an $\mathscr{F}$ - stopping time τ_i

and an $\mathscr{F}_{\tau_i-}$ - measurable intervention ζ_i is called an impulse control. The admissible control set, denoted as $\mathscr{I}$, is the set of all N-tuples $(\tau, \zeta) = \{(\tau_i, \zeta_i)\}_{i=1}^N$ of such impulse controls.

Over the entire time horizon $[0, T]$, the process $X(\cdot)$ can be written as

$$X(t) = X(0) + \int_0^t b(s, X(s); \theta(s))ds + \int_0^t \sigma(s, X(s))dW(s) + \sum_{\tau_i \le t} \gamma(X(\tau_i-), \zeta_i), \ 0 \le t \le T. \tag{3.3}$$

3.2 General Theory

This subsection solves the problem of choosing an optimal $(\tau^*, \zeta^*) = \{(\tau_i^*, \zeta_i^*)\}_{i=1}^N$ over all admissible impulse controls $(\tau, \zeta) = \{(\tau_i, \zeta_i)\}_{i=1}^N$ in $\mathscr{I}$ to achieve the maximal expected reward

$$\sup_{(\tau,\zeta)\in\mathscr{I}} \mathbb{E}\left[\int_0^T h(X(t))dt + \xi(X(T)) + \sum_{\tau_i \le T} c(X(\tau_i-), \zeta_i)\right]. \tag{3.4}$$

The rewards ξ and $h : \mathbb{R} \to \mathbb{R}$ are deterministic measurable functions satisfying Assumption 2.1 (3)-(4). Besides, we impose polynomial growth condition on the deterministic measurable functions γ and $c : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ in the state variable.

Assumption 3.1 *(1) The functions $\gamma(x, z)$ and $c(x, z)$ have polynomial growth rates in $x \in \mathbb{R}$, uniformly for all $z \in \mathbb{R}$.*

Let $\mathscr{G} = \{\mathscr{G}\}_{0\le t\le T}$ be the filtration defined as in (2.15), with $X(\cdot)$ defined in (3.2). The continuous part of the state process $X(\cdot)$, denoted by

$$\begin{aligned} I(t) :=& X(t) - \sum_{\tau_i \le t} \gamma(X(\tau_i-), \zeta_i) \\ =& x_0 + \int_0^t b(s, X(s); \theta(s))ds + \int_0^t \sigma(s, X(s))dW(s), \ 0 \le t \le T, \end{aligned} \tag{3.5}$$

is an $\mathscr{F}$-adapted process. Construct a probability measure $\mathbb{P}_0$, under which the $\mathscr{F}$-adapted process

$$W^0(t) := \int_0^t \sigma^{-1}(s, X(s))dI(s), \ 0 \le t \le T \tag{3.6}$$

is a standard Brownian motion. Then the process

$$I(t) = x_0 + \int_0^t \sigma(s, X(s))dW^0(s), \ 0 \le t \le T \tag{3.7}$$

is a continuous local $(\mathbb{P}_0, \mathscr{F})$-martingale with its quadratic variation process

$$\langle I \rangle_t = \int_0^t \sigma^2(s, X(s))ds, \ 0 \le t \le T. \tag{3.8}$$

In the probability space $(\Omega, \mathscr{G}, \mathbb{P}_0)$, the random variables ρ and U are independent, and the random vector (ρ, U) is independent of the filtration $\mathscr{F}$, thus independent of the processes $X(\cdot)$ and $I(\cdot)$. We assign the independent random variables ρ and U the same $\mathbb{P}_0$-prior distributions mandated by (2.4) and (2.5).

Define the $\mathscr{G}$-adapted Radon-Nikodym derivative process by

$$Z(t) = \exp\left\{\int_0^t \frac{b(s, X(s); \theta(s))}{\sigma^2(s, X(s))} dI(s) - \frac{1}{2}\int_0^t \frac{b^2(s, X(s); \theta(s))}{\sigma^2(s, X(s))} ds\right\},\ 0 \le t \le T. \tag{3.9}$$

With this Radon-Nikodym derivative $Z(\cdot)$, and via the expression (2.21), define the probability measure $\tilde{\mathbb{P}}$ which is equivalent to $\mathbb{P}_0$. Under the probability measure $\tilde{\mathbb{P}}$, the random variables ρ and U remains independent with prior distributions (2.4) and (2.5). Again by the change of measure for local martingales in [15] Van Schuppen and Wong (1974), the process

$$\left\{I(t) - \int_0^t b(s, X(s); \theta(s)) ds\right\}_{0 \le t \le T} \tag{3.10}$$

is a local $(\tilde{\mathbb{P}}, \mathscr{G})$-martingale, also having the instantaneous quadratic variation $\sigma^2(\cdot, X(\cdot))$. The process $\tilde{W}(\cdot)$ defined as

$$\tilde{W}(t) := \int_0^t \sigma^{-1}(s, X(s)) dI(s) - \int_0^t \sigma^{-1}(s, X(s)) b(s, X(s); \theta(s)) ds,\ 0 \le t \le T \tag{3.11}$$

is a continuous local $(\tilde{\mathbb{P}}, \mathscr{G})$-martingale with quadratic variation t, thus a standard $\tilde{\mathbb{P}}$-Brownian motion. The process $X(\cdot)$ satisfies the equation (3.3) with the $\mathbb{P}$-Brownian motion $W(\cdot)$ replaced by the $\tilde{\mathbb{P}}$-Brownian motion $\tilde{W}(\cdot)$. It then follows from the pathwise uniquess of the solution to equation (3.3) that $X(\cdot)$ has the same distribution under the probability measures $\tilde{\mathbb{P}}$ and $\mathbb{P}$.

The change of measure, once again, transforms optimization under the measure $\mathbb{P}$ into optimization under the measure $\mathbb{P}_0$, by

$$\begin{aligned}
&\sup_{(\tau,\zeta)\in\mathscr{I}} \mathbb{E}\left[\int_0^T h(X(t))dt + \xi(X(T)) + \sum_{\tau_i \le T} c(X(\tau_i -), \zeta_i)\right] \\
&= \sup_{(\tau,\zeta)\in\mathscr{I}} \tilde{\mathbb{E}}\left[\int_0^T h(X(t))dt + \xi(X(T)) + \sum_{\tau_i \le T} c(X(\tau_i -), \zeta_i)\right] \\
&= \sup_{(\tau,\zeta)\in\mathscr{I}} \mathbb{E}_0\left[Z(T)\left(\int_0^T h(X(t))dt + \xi(X(T)) + \sum_{\tau_i \le T} c(X(\tau_i -), \zeta_i)\right)\right] \\
&= \sup_{(\tau,\zeta)\in\mathscr{I}} \mathbb{E}_0\left[\mathbb{E}_0\left[Z(T)\,|\mathscr{F}_T\right]\left(\int_0^T h(X(t))dt + \xi(X(T)) + \sum_{\tau_i \le T} c(X(\tau_i -), \zeta_i)\right)\right].
\end{aligned} \tag{3.12}$$

For every number $u \in \Theta$, we introduce the $\mathscr{F}$-adapted likelihood ratio process

$$L(t; u) = \exp\left\{\int_0^t \frac{b(s, X(s); u)}{\sigma^2(s, X(s))} dI(s) - \frac{1}{2}\int_0^t \frac{b^2(s, X(s); u)}{\sigma^2(s, X(s))} ds\right\},\ 0 \le t \le T, \tag{3.13}$$

a continuous process satisfying the stochastic integral equation

$$L(t;u) = \int_0^t L(s;u) \frac{b(s,X(s);u)}{\sigma^2(s,X(s))} dI(s),\ 0 \le t \le T. \tag{3.14}$$

The Radon-Nikodym derivative $Z(\cdot)$ can be written in terms of the likelihood ratio $L(\cdot;u)$ and the random variables ρ and U as in (2.20). Because of the independence of the observations $X(\cdot)$ and the prior distributions of ρ and U under the reference measure $\mathbb{P}_0$, the posterior $\mathbb{P}_0$-expectation of $Z(\cdot)$ is the continuous process $\mathbb{E}_0\left[Z(t)\,|\mathscr{F}_t\right]$ in the form of (2.25). Applying Itô's formula to $\mathbb{E}_0\left[Z(t)\,|\mathscr{F}_t\right]$, and using (2.28) and (3.14), we get, for $0 \le t \le T$,

$$\begin{aligned}
&d\left(\mathbb{E}_0\left[Z(t)\,|\mathscr{F}_t\right]\right) \\
=&\left(\sum_{j=1}^m p_j \left(\int_0^t \frac{L(s;\mu_0)}{L(s;\mu_j)} \lambda e^{-\lambda s} ds\right) L(t;\mu_j) \frac{b(t,X(t);\mu_j)}{\sigma^2(t,X(t))} + e^{-\lambda t} L(t;u) \frac{b(t,X(t);\mu_0)}{\sigma^2(t,X(t))}\right) dI(t).
\end{aligned} \tag{3.15}$$

Both the likelihood ratio process $L(\cdot;u)$ and the posterior expectation process $\{\mathbb{E}_0\left[Z(t)\,|\mathscr{F}_t\right]\}_{0\le t\le T}$ of the Radon-Nikodym derivative are local $(\mathbb{P}_0, \mathscr{G})$-martingales.

Lemma 3.1 *For $0 \le t \le T$, $x \in \mathbb{R}^d$, $l = (l_0, l_1, \cdots, l_m) \in \mathbb{R}^{m+1}$, $r = (r_1, \cdots, r_m) \in \mathbb{R}^m$, and $z \in \mathbb{R}$, let the function α be as in (2.31), and define the function β as*

$$\begin{aligned}
&\beta(t,x,l,r,z) \\
=&\left(\sum_{j=1}^m p_j l_j r_j + e^{-\lambda t} l_0\right) \left(\xi(x+\gamma(x,z)) - \xi(x) + \xi'(x)\gamma(x,z) + c(x,z)\right).
\end{aligned} \tag{3.16}$$

Then, for $0 \le t \le T$, we have

$$\begin{aligned}
&\mathbb{E}_0\left[Z(t)\,|\mathscr{F}_t\right] \left(\int_0^t h(X(s))ds + \xi(X(t))\right) \\
=&M_0(t) + \int_0^t \alpha\left(s, X(s), L(s), R(s)\right) ds + \sum_{\tau_i \le t} \beta(\tau_i, X(\tau_i-), L(\tau_i-), R(\tau_i-), \zeta_i),
\end{aligned} \tag{3.17}$$

where $M_0(\cdot)$ is some square integrable $(\mathbb{P}_0, \mathscr{F})$-martingale with $M_0(0) = \xi(X(0))$, and the processes $L(\cdot)$ and $R(\cdot)$ are defined as in (3.13), (2.35), (2.33) and (2.34).

Proof. Applying Itô's formula for semimartingales with jumps,

$$\begin{aligned}
&\mathbb{E}_0\left[Z(t)\,|\mathscr{F}_t\right] \left(\int_0^t h(X(s))ds + \xi(X(t)) + \sum_{\tau_i \le t} c(X(\tau_i-), \zeta_i)\right) \\
=&\xi(X(0)) + \int_0^t \left(\int_0^{s-} h(X(u))du + \xi(X(s-)) + \sum_{\tau_i \le s-} c(X(\tau_i-), \zeta_i)\right) d\mathbb{E}_0\left[Z(s)\,|\mathscr{F}_s\right] \\
&+ \int_0^t \mathbb{E}_0\left[Z(s-)\,|\mathscr{F}_{s-}\right] \xi'(X(s-)) dI(s) \\
&+ \int_{0+}^t \alpha\left(s-, X(s-), L(s-), R(s-)\right) ds + \sum_{\tau_i \le t} \beta(\tau_i, X(\tau_i-), L(\tau_i-), R(\tau_i-), \zeta_i).
\end{aligned} \tag{3.18}$$

By change of variables and the continuity of Riemann integrals,

$$
\begin{aligned}
&\int_{0+}^{t} \alpha\left(s-, X(s-), L(s-), R(s-)\right) ds \\
=&\int_{0}^{t-} \alpha\left(s, X(s), L(s), R(s)\right) ds \\
=&\int_{0}^{t} \alpha\left(s, X(s), L(s), R(s)\right) ds.
\end{aligned} \tag{3.19}
$$

Define

$$
\begin{aligned}
M_0(t) :=&\xi(X(0)) + \int_0^t \left(\int_0^{s-} h(X(u))du + \xi(X(s-)) + \sum_{\tau_i \le s-} c(X(\tau_i-), \zeta_i)\right) d\mathbb{E}_0\left[Z(s)\,|\mathscr{F}_s\right] \\
&+ \int_0^t \mathbb{E}_0\left[Z(s-)\,|\mathscr{F}_{s-}\right] \xi'(X(s-))dI(s).
\end{aligned} \tag{3.20}
$$

From (3.15),

$$
\begin{aligned}
&M_0(t) := \xi(X(0)) + \\
&\int_0^t \left(\mathbb{E}_0\left[Z(s-)\,|\mathscr{F}_{s-}\right]\xi'(X(s-)) + \left(\int_0^{s-} h(X(u))du + \xi(X(s-)) + \sum_{\tau_i \le s-} c(X(\tau_i-), \zeta_i)\right) \cdot \right. \\
&\left. \left(\sum_{j=1}^m p_j \left(\int_0^t \frac{L(s;\mu_0)}{L(s;\mu_j)} \lambda e^{-\lambda s} ds\right) L(t;\mu_j) \frac{b(t, X(t);\mu_j)}{\sigma^2(t, X(t))} + e^{-\lambda t} L(t;u) \frac{b(t, X(t);\mu_0)}{\sigma^2(t, X(t))}\right)\right) dI(s),
\end{aligned} \tag{3.21}
$$

an integral of $\mathbb{P}_0$-square integrable processes with respect to the local $(\mathbb{P}_0, \mathscr{F})$-martingale $I(\cdot)$, hence $M_0(\cdot)$ is also a local $(\mathbb{P}_0, \mathscr{F})$-martingale.
The proof of $\mathbb{P}_0$-uniform integrability of $\{M_0(\tau)\}_{\tau \in \mathscr{T}}$ is similar to that in the proof of Lemma 2.2.

□

Lemma 3.2 *For any impulse control* $(\tau, \zeta) \in \mathscr{I}$,

$$
\begin{aligned}
&\mathbb{E}_0\left[\mathbb{E}_0\left[Z(T)\,|\mathscr{F}_T\right]\left(\int_0^T h(X(t))dt + \xi(X(T)) + \sum_{\tau_i \le T} c(X(\tau_i-), \zeta_i)\right)\right] \\
=&\xi(X(0)) + \mathbb{E}_0\left[\int_0^T \alpha\left(s, X(s), L(s), R(s)\right) ds + \sum_{\tau_i \le T} \beta(\tau_i, X(\tau_i-), L(\tau_i-), R(\tau_i-), \zeta_i)\right].
\end{aligned} \tag{3.22}
$$

Proof. This result follows from Lemma 3.1. □

Lemma 3.3 *The triple* $(X(\cdot), L(\cdot), R(\cdot))$ *is a* $(2m+2)$*-dimensional Markov process on every time interval* $[\tau_i, \tau_{i+1})$*, for* $i = 0, 1, \cdots, N-1$.

Proof. For $t \in (\tau_i, \tau_{i+1})$, the state process $X(t)$ is continuous, so $dI(t) = dX(t)$, where $I(\cdot)$ is the continuous part defined in (3.5). The triple $(X(\cdot), L(\cdot), R(\cdot))$ is then the pathwise unique, strong solution to the SDE (2.45) with the initial value $(X(\tau_i), L(\tau_i), R(\tau_i))$ at the time τ_i. The Markovian property follows from Lemma 2.4. □

Let $\mathscr{I}^M$ be the collection of Markovian impulse controls $(\tau, \zeta) = \{(\tau_i, \zeta_i)\}_{i=1}^N$, where the interventions satisfy

$$\zeta_i = \bar{\zeta}_i(\tau_i, X(\tau_i-), L(\tau_i), R(\tau_i)), \tag{3.23}$$

for some deterministic measurable function $\bar{\zeta}_i : [0, T] \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^{m+1}$.

Theorem 3.1 *The impulse control problem (3.4) is equivalent to a non-stationary Markovian impulse control problem under measure $\mathbb{P}_0$ with respect to the filtration $\mathscr{F}$, in the sense that*

$$\begin{aligned}
&\sup_{(\tau,\zeta)\in\mathscr{I}} \mathbb{E}\left[\int_0^T h(X(t))dt + \xi(X(T)) + \sum_{\tau_i\le T} c(X(\tau_i-), \zeta_i)\right] \\
=&\xi(X(0)) + \sup_{(\tau,\zeta)\in\mathscr{I}} \mathbb{E}_0\left[\int_0^T \alpha\left(s, X(s), L(s), R(s)\right) ds + \sum_{\tau_i\le T} \beta(\tau_i, X(\tau_i-), L(\tau_i-), R(\tau_i-), \zeta_i)\right] \\
=&\xi(X(0)) + \sup_{(\tau,\zeta)\in\mathscr{I}^M} \mathbb{E}_0\left[\int_0^T \alpha\left(s, X(s), L(s), R(s)\right) ds + \sum_{\tau_i\le T} \beta(\tau_i, X(\tau_i-), L(\tau_i-), R(\tau_i-), \zeta_i)\right].
\end{aligned} \tag{3.24}$$

Proof. The first equality comes from identity (3.12), Lemma 3.2. By Lemma 3.3, the augmented state process $(X(\cdot), L(\cdot), R(\cdot))$ is Markovian between any two intervention times τ_i and τ_{i+1}, then the supremum over all admissible controls is achieved among the Markovian controls (c.f. section 1.4 of [11] Krylov (1980) and section 3.1 of [12] Øksendal and Sulem (2007)). □

The rest of this subsection will characterize the value function (3.24) as viscosity solutions to a chain of N interconnected variational inequalities. The impulse control problem was studied by Bensoussan and Lions in [1], [2], [3] and [4] in the 1970's. Again, due to the degeneracy of the coefficients of the second order terms of the variational inequalities (3.29) and (3.30), it will not be handy to demonstrate the existence and regularity of their solutions, which is required by the verification theorem approach. Following routine arguments using the dynamic programming principle, the value function (3.24) can be shown to solve the variational inequalities (3.29) and (3.30) in the viscosity sense.

Lemma 3.4 *(Dynamic Programming Principle)*
For any $k \in \{1, 2, \cdots, N\}$, and any $0 \le t \le T$, let $\mathscr{I}_{t,k}$ be the set of admissible interventions $\{(\tau_{N-k+1}, \zeta_{N-k+1}), (\tau_{N-k+2}, \zeta_{N-k+2}), \cdots, (\tau_N, \zeta_N)\}$ such that $\tau_{N-k+1} \ge t$. There exists a deterministic measurable function $v_0, v_1, \cdots, v_N : [0, T] \times \mathbb{R} \times (0, \infty)^{m+1} \times [0, \infty)^m \to \mathbb{R}$,

such that

$$
\begin{aligned}
&v_k(t,x,l,r)\\
=&\sup_{\{(\tau_i,\zeta_i)\}_{i=N-k+1}^N\in\mathscr{I}_{t,k}}\mathbb{E}_0\left[\int_t^T a\left(s,X(s),L(s),R(s)\right)ds+\sum_{i=N-k+1}^N\beta(\tau_i,X(\tau_i-),L(\tau_i-),R(\tau_i-),\zeta_i)\middle|\mathscr{F}_t\right],
\end{aligned}
\tag{3.25}
$$

for $k=1,\cdots,N$, and

$$
v_0(t,x,l,r)=\mathbb{E}_0\left[\int_t^T\alpha\left(s,X(s),L(s),R(s)\right)ds\middle|\mathscr{F}_t\right]. \tag{3.26}
$$

The the value functions $v_1,\cdots,v_N$ satisfy the dynamic programming principle

$$
\begin{aligned}
&v_k(t,x,l,r)\\
=&\sup_{(\tau_{N-k+1},\zeta_{N-k+1})\in\mathscr{I}_{t,1}}\mathbb{E}_0\Bigg[\int_t^{\tau_{N-k+1}}\alpha\left(s,X(s),L(s),R(s)\right)ds\\
&+\beta(\tau_{N-k+1},X(\tau_{N-k+1}-),L(\tau_{N-k+1}-),R(\tau_{N-k+1}-),\zeta_{N-k+1})\\
&+v_{k-1}(\tau_{N-k+1},X(\tau_{N-k+1}),L(\tau_{N-k+1}),R(\tau_{N-k+1}),\zeta_{N-k+1})\Bigg|\mathscr{F}_t\Bigg].
\end{aligned}
\tag{3.27}
$$

Theorem 3.2 *For any function $f:[0,T]\times\mathbb{R}\times(0,\infty)^{m+1}\times[0,\infty)^m\to\mathbb{R}$, define a mapping $\mathscr{M}$ by*

$$
\left(\mathscr{M}f\right)(t,x,l,r):=\sup_{z\in\mathbb{R}}\{f(t,x+\gamma(x,z),l,r)+\beta(t,x,l,r,z)\}, \tag{3.28}
$$

for all $(t,x,l,r)\in[0,T]\times\mathbb{R}\times(0,\infty)^{m+1}\times[0,\infty)^m$, then the value functions $v_0,v_1,\cdots,v_N$ defined in Lemma 3.4 are viscosity solutions (Definition 4.1 (2)) to the variational inequalities

$$
\min\left\{-\left(\frac{\partial}{\partial t}v_k+\mathscr{A}v_k+\alpha\right)(t,x,l,r),\left(v_k-\mathscr{M}v_{k-1}\right)(t,x,l,r)\right\}=0, \tag{3.29}
$$

for $k=1,2,\cdots,N$, and

$$
\left(\frac{\partial}{\partial t}v_0+\mathscr{A}v_0+\alpha\right)(t,x,l,r)=0, \tag{3.30}
$$

with the terminal condition

$$
v_0(T,x,l,r)=0, \tag{3.31}
$$

for all $0\le t\le T$ and $(x,l,r)\in\mathbb{R}\times(0,\infty)^{m+1}\times[0,\infty)^m$. The infinitesimal generator $\mathscr{A}$ is defined in (2.49).

3.3 Computing the Value Function

With the notations inherited from section 2.3, the coefficients b_Y and σ_Y are again locally Lipschitz but not necessarily globally Lipschitz, so the solutions to the variational inequalities (3.29) and (3.30) with the boundary condition (3.31) are not necessarily unique. Like in section 2.3, we need to approximate the set of solutions which provide the value functions (3.25) and (3.26). To get uniqueness of solutions to the variational inequalities associated to the approximating sequence of impulse control problems, in addition to Assumption 3.1 (1), the functions γ and c are assumed locally Lipschitz.

Assumption 3.1 *(continued)*
(2) For every compact subset $K_n \subset \mathbb{R}\times\mathbb{R}^{m+1}\times\mathbb{R}^m$, there exists a constant $C_n > 0$, uniformly for all $0 \le t \le T$ and for all deterministic measurable functions $\bar{\zeta} : [0,T] \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^{m+1}$, such that

$$\begin{aligned}&|\gamma(x^1, \bar{\zeta}(t, x^1, l^1, r^1)) - \gamma(x^2, \bar{\zeta}(t, x^2, l^2, r^2))| + |c(x^1, \bar{\zeta}(t, x^1, l^1, r^1)) - c(x^2, \bar{\zeta}(t, x^2, l^2, r^2))|\\ \le& C_n(|x^1 - x^2| + |l^1 - l^2| + |r^1 - r^2|),\end{aligned} \tag{3.32}$$

for all (x^1, l^1, r^1), $(x^2, l^2, r^2) \in K_n$.

The following theorems describe the approximating scheme to compute the value functions (3.25) and (3.26).

Theorem 3.3 *There exists a function $v_k^n : [0,T] \times \bar{\mathscr{O}}_n \to \mathbb{R}$, such that*

$$v_k^n(t, Y(t)) = \sup_{\{(\tau_i,\zeta_i)\}_{i=N-k+1}^N \in \mathscr{I}_{t,k},\ \tau_N \le T_n} \mathbb{E}_0 \left[\int_t^{T_n} \alpha\left(s, Y(s)\right) ds + \sum_{i=N-k+1}^{N} \beta(\tau_i, Y(\tau_i-), \zeta_i) \middle| \mathscr{F}_t \right], \tag{3.33}$$

$0 \le t \le T$, for $k = 1, \cdots, N$, and

$$v_0^n(t, x, l, r) = \mathbb{E}_0 \left[\int_t^{T_n} \alpha\left(s, Y(s)\right) ds \middle| \mathscr{F}_t \right]. \tag{3.34}$$

The value functions $v_0^n, v_1^n, \cdots, v_N^n$ are viscosity solutions to the variational inequalities

$$\min\left\{ -\left(\frac{\partial}{\partial t} v_n + \mathscr{A} v_n + \alpha\right)(t, y), \left(v_n + \alpha\right)(t, y) \right\} = 0,\ (t, y) \in Q_n \setminus \partial^* Q_n, \tag{3.35}$$

for $k = 1, \cdots, N$, and

$$\left(\frac{\partial}{\partial t} v_0^n + \mathscr{A} v_0^n + \alpha\right)(t, y) = 0,\ (t, y) \in Q_n \setminus \partial^* Q_n, \tag{3.36}$$

with the boundary condition

$$v_0^n(t, y) = 0,\ (t, y) \in \partial^* Q_n. \tag{3.37}$$

As $n \to \infty$, the value function v_k^n converges pointwise to the value function v_k defined as in (3.25) and (3.26), for $k = 0, 1, \cdots, N$.

Lemma 3.5 *For $k = 0, 1, \cdots, N$, the value functions v_k^n as in (3.33) and (3.34) are Lipschitz over the bounded domain $\bar{\mathscr{O}}_n$, uniformly for all $0 \le t \le T$, meaning that there exists a constant $C_n > 0$, such that*

$$|v_k^n(t, y^1) - v_k^n(t, y^2)| \le C_n |y^1 - y^2|, \text{ for all } (t, y^1), (t, y^2) \in [0, T] \times \bar{\mathscr{O}}_n. \tag{3.38}$$

Proof. The same as proving Lemma 2.6.

Theorem 3.4 *Suppose $\underline{v}_0^n, \underline{v}_1^n, \cdots, \underline{v}_N^n$ and $\bar{v}_0^n, \bar{v}_1^n, \cdots, \bar{v}_N^n$ in $\mathscr{C}_{Lip}([0, T] \times \bar{\mathscr{O}}_n)$ are, respectively, viscosity subsolutions and viscosity supersolutions to the variational inequalities (3.35) and (3.36) with the boundary condition (3.37), then the comparison result*

$$\sup_{\bar{Q}_n} (\underline{v}_k^n - \bar{v}_k^n) = \sup_{\partial^* Q_n} (\underline{v}_k^n - \bar{v}_k^n) = 0 \tag{3.39}$$

holds for all $k = 0, 1, \cdots, N$, hence the viscosity solutions $v_0^n, v_1^n, \cdots, v_N^n$ to (3.35)-(3.37) are unique.

Proof. Starting from $k = 0$, inductively apply Theorem 2.4 to $\underline{v}_k^n$ and $\bar{v}_k^n$, for $k = 0, 1, \cdots, N$.

Proposition 3.1 *(iterative procedure for optimization) For every $k = 1, 2, \cdots, N$, iteratively define an $\mathscr{F}$-stopping time*

$$\tau_k^* := \inf \left\{ \tau_{k-1}^* < t \le T \,|\, v_{N-k+1}(t, X(t), L(t), R(t)) \le \mathscr{M} v_{N-k}(t, X(t), L(t), R(t)) \right\}, \tag{3.40}$$

with the convention that $\tau_0^* = 0$. For every $\epsilon \ge 0$, let $z_k(t, x, l, r; \epsilon)$ be any of the real numbers such that

$$0 \le \sup_z \{ v_{N+1-k}(t, x + \gamma(x, z), l, r) + \beta(t, x, l, r, z) \} - v_{N-k}(t, x, l, r; \epsilon) \le \frac{\epsilon}{N}, \tag{3.41}$$

and define an $\mathscr{F}_{\tau_k^*-}$-measurable random variable

$$\zeta_k^*(\epsilon) := z_k (\tau_k^*, X(\tau_k^*-), L(\tau_k^*), R(\tau_k^*); \epsilon). \tag{3.42}$$

The set of impulse controls $\{\tau_k^*, \zeta_k^*(\epsilon)\}_{k=1}^N$ in $\mathscr{I}$ is ϵ-optimal, in the sense that

$$\begin{aligned} &\mathbb{E}_0 \left[\int_0^T \alpha(s, X(s), L(s), R(s)) \, ds + \sum_{k=1}^N \beta(\tau_k^*, X(\tau_k^*-), L(\tau_k^*-), R(\tau_k^*-), \zeta_k^*(\epsilon)) \right] \\ &\ge \sup_{(\tau,\zeta) \in \mathscr{I}} \mathbb{E}_0 \left[\int_0^T \alpha(s, X(s), L(s), R(s)) \, ds + \sum_{i=1}^N \beta(\tau_i, X(\tau_i-), L(\tau_i-), R(\tau_i-), \zeta_i) \right] - \epsilon. \end{aligned} \tag{3.43}$$

If the real numbers $\{z_1(t, x, l, r; 0), z_2(t, x, l, r; 0), \cdots, z_N(t, x, l, r; 0)\}$ exist, meaning that the supremum in (3.41) can be achieved, then the set of impulse controls $\{\tau_k^*, \zeta_k^*(0)\}_{k=1}^N$ in $\mathscr{I}$ achieve the supremum in (3.24).

3.4 Examples

Example 3.1 *(Geometric Brownian motion with drift uncertainty, continued)*
Consider the very geometric Brownian motion model in Example 2.4. Choose stopping times $0 \leq \tau_1^* \leq \tau_2^* \leq T$ in $\mathscr{T}$, stopping times of the filtration $\mathscr{F}$ generated by the geometric Brownian motion $X(\cdot)$, to achieve the supremum in

$$\sup_{\tau_1 \text{ and } \tau_2 \in \mathscr{T},\ \tau_1 \leq \tau_2} \mathbb{E}\left[X(\tau_2) - X(\tau_1)\right]. \tag{3.44}$$

Suppose $X(\cdot)$ is the price process of a certain stock, then, with zero interest rate and in the absence of transaction cost, the value (3.44) is the best possible profit from first buying and then selling one share of this stock, observing the price evolution only.

Solution. Comparing the SDEs (2.11) with (3.3), and the goals (3.4) with (3.44), we are trying to solve the impulse control problem with $\gamma = 0$, $h(\cdot) = 0$, $\xi(\cdot) = 0$, $c(x, z) = zx$ for $x \in \mathbb{R}^n$ and $z \in \mathbb{R}$, $\zeta_1 = -1$ and $\zeta_2 = 1$.
When the interventions have no effect on the state process $X(\cdot)$, the change of measure from $\mathbb{P}$ to $\mathbb{P}_0$ reduces to the optimal stopping case in section 2. Here in this example, the change of measure is exactly the same as that in (2.87) for Example 2.4. The functions $\alpha = 0$ and β takes the form

$$\beta(t, x, z, l, r) = zx\left(\sum_{j=1}^{m} p_j l_j r_j + e^{-\lambda t} l_0\right). \tag{3.45}$$

By Theorem 3.1,

$$\begin{aligned} &\sup_{\tau_1 \text{ and } \tau_2 \in \mathscr{T},\ \tau_1 \leq \tau_2} \mathbb{E}\left[X(\tau_2) - X(\tau_1)\right] \\ = &\sup_{\tau_1 \text{ and } \tau_2 \in \mathscr{T},\ \tau_1 \leq \tau_2} \mathbb{E}_0\left[X(\tau_2)\left(\sum_{j=1}^{m} p_j L(\tau_2; \mu_j) R(\tau_2; \mu_j) + e^{-\lambda t} L(\tau_2; \mu_0)\right)\right. \\ &\qquad\qquad \left. - X(\tau_1)\left(\sum_{j=1}^{m} p_j L(\tau_1; \mu_j) R(\tau_1; \mu_j) + e^{-\lambda t} L(\tau_1; \mu_0)\right)\right]. \end{aligned} \tag{3.46}$$

There exist deterministic measurable functions v_1 and $v_2 : [0, T] \times (0, \infty) \times (0, \infty)^{m+1} \times [0, \infty)^m$, such that

$$v_1(t, X(t), L(t), R(t)) = \sup_{\tau_2 \in \mathscr{S}_t} \mathbb{E}\left[\beta(\tau_2, X(\tau_2), 1, L(\tau_2), R(\tau_2))\,|\mathscr{F}_t\right], \tag{3.47}$$

and

$$v_2(t, X(t), L(t), R(t)) = \sup_{\tau_1 \in \mathscr{S}_t} \mathbb{E}\left[\beta(\tau_1, X(\tau_1), -1, L(\tau_1), R(\tau_1)) + v_1(\tau_1, X(\tau_1), L(\tau_1), R(\tau_1))\,|\mathscr{F}_t\right]. \tag{3.48}$$

The functions v_1 and v_2 are viscosity solutions to the variational inequalities

$$\min\left\{-\left(\frac{\partial}{\partial t} v_2 + \mathscr{A} v_2\right)(t, x, l, r), (v_2 - v_1)(t, x, l, r) + x\left(\sum_{j=1}^{m} p_j l_j r_j + e^{-\lambda t} l_0\right)\right\} = 0, \tag{3.49}$$

and

$$\min\left\{-\left(\frac{\partial}{\partial t}v_1+\mathscr{A}v_1\right)(t,x,l,r), v_1(t,x,l,r)-x\left(\sum_{j=1}^{m}p_jl_jr_j+e^{-\lambda t}l_0\right)\right\}=0, \tag{3.50}$$

with terminal condition

$$v_1(T,x,l,r)=x\left(\sum_{j=1}^{m}p_jl_jr_j+e^{-\lambda T}l_0\right), \tag{3.51}$$

for $0\leq t\leq T$, $(x,l,r)\in(0,\infty)\times(0,\infty)^m\times[0,\infty)^{m+1}$. The infinitesimal generator $\mathscr{A}$ is defined as in (2.91). We may reduce the dimensionality as in Example 2.4. The variational inequalities can be solved via the approximation scheme suggested in section 3.3. The optimal stopping times

$$\begin{aligned}\tau_1^*=\inf\Bigg\{0\leq t\leq T\,\Bigg|\,&(v_2-v_1)(t,X(t),L(t),R(t))\\&+X(t)\left(\sum_{j=1}^{m}p_jL(t;\mu_j)R(t;\mu_j)+e^{-\lambda t}L(t;\mu_0)\right)\leq 0\Bigg\},\end{aligned} \tag{3.52}$$

and

$$\begin{aligned}\tau_2^*=\inf\Bigg\{\tau_1^*\leq t\leq T\,\Bigg|\,&v_1(t,X(t),L(t),R(t))\\&-X(t)\left(\sum_{j=1}^{m}p_jL(t;\mu_j)R(t;\mu_j)+e^{-\lambda t}L(t;\mu_0)\right)\leq 0\Bigg\}\end{aligned} \tag{3.53}$$

achieve the supremum in (3.44), and the optimal value of the round-way transaction can be computed from

$$\sup_{\tau_1 \text{ and } \tau_2\in\mathscr{T},\ \tau_1\leq\tau_2}\mathbb{E}\left[X(\tau_2)-X(\tau_1)\right]=v_2(0,X(0),\mathbf{1},\mathbf{0}). \tag{3.54}$$

□

4 Appendix

Proof of Lemma 2.1. Taking an arbitrary $\mathscr{F}$-stopping time τ with values in $[0,T]$, by Holder's inequalities,

$$\begin{aligned}&\mathbb{E}_0\left[L^{n_1}(\tau;\mu_j)R^{n_2}(\tau;\mu_j)\sup_{0\le t\le T}|X(t)|^{n_3}\right]\\&\le\left(\mathbb{E}_0\left[L^{4n_1}(\tau;\mu_j)\right]\right)^{1/4}\left(\mathbb{E}_0\left[R^{4n_2}(\tau;\mu_j)\right]\right)^{1/4}\left(\mathbb{E}_0\left[\sup_{0\le t\le T}|X(t)|^{2n_3}\right]\right)^{1/2}.\end{aligned}\tag{4.1}$$

By the definition of the likelihood ratio in (2.19), for any $u\in\Theta$, and any $n_1=1,2,\cdots$,

$$\begin{aligned}L^{4n_1}(t;u)=&\exp\left\{4n_1\int_0^t\frac{b(s,X(s);u)}{\sigma^2(s,X(s))}dX(s)-8{n_1}^2\int_0^t\frac{b^2(s,X(s);u)}{\sigma^2(s,X(s))}ds\right\}\\&\cdot\exp\left\{(8n_1^2-2n_1)\int_0^t\frac{b^2(s,X(s);u)}{\sigma^2(s,X(s))}ds\right\},\ 0\le t\le T,\end{aligned}\tag{4.2}$$

the multiplicand of which is a $(\mathbb{P}_0,\mathscr{F})$-supermartingale with initial value one, and the multiplicator bounded by $\exp\{(2n_1^2-n_1)TC^2\}$, so by optional sampling theorem,

$$\mathbb{E}_0\left[L^{4n_1}(\tau;\mu_j)\right]\le\exp\left\{(8n_1^2-2n_1)TC^2\right\}.\tag{4.3}$$

Similarly, for $n_2=1,2,\cdots$,

$$\begin{aligned}&\frac{L^{4n_2}(t;\mu_0)}{L^{4n_2}(t;\mu_j)}\\&=\exp\left\{4n_2\int_0^t\frac{b(s,X(s);\mu_0)-b(s,X(s);\mu_j)}{\sigma^2(s,X(s))}dX(s)-8{n_2}^2\int_0^t\frac{(b(s,X(s);\mu_0)-b(s,X(s);\mu_j))^2}{\sigma^2(s,X(s))}ds\right\}\\&\quad\cdot\exp\left\{\int_0^t\frac{(16{n_2}^2-2n_2)b^2(s,X(s);\mu_0)+(16{n_2}^2+2n_2)b^2(s,X(s);\mu_j)}{\sigma^2(s,X(s))}ds\right\},\ 0\le t\le T\end{aligned}\tag{4.4}$$

is the product of a $(\mathbb{P}_0,\mathscr{F})$-supermartingale and a bounded multiplicator, so

$$\mathbb{E}_0\left[\frac{L^4(t;\mu_0)}{L^4(t;\mu_j)}\right]\le\exp\left\{32{n_2}^2TC^2\right\}.\tag{4.5}$$

Then,

$$\mathbb{E}_0\left[R^{4n_2}(\tau;\mu_j)\right]\le\lambda^{4n_2}T^{4n_2-1}\int_0^T\mathbb{E}_0\left[\frac{L^4(t;\mu_0)}{L^4(t;\mu_j)}\right]dt\le\lambda^{4n_2}T^{4n_2}\exp\left\{32{n_2}^2TC^2\right\}.\tag{4.6}$$

Since $X(\cdot)$ is a local $\mathbb{P}_0$-martingale, from Burkholder-Davis-Gundy inequality (e.g. page 166 of [8] Karatzas and Shreve (1988)), we have, for $n_3=1,2,\cdots$,

$$\mathbb{E}_0\left[\sup_{0\le t\le T}|X(t)|^{2n_3}\right]\le C_{n_3}\mathbb{E}\left[<X>_T^{n_3}\right],\tag{4.7}$$

for some constant $0 \le C_{n_3} < \infty$. But

$$
\begin{aligned}
&\mathbb{E}_0\left[< X >_T^{n_3}\right] = \mathbb{E}_0\left[\left(\int_0^T \sigma^2(t, X(t))dt\right)^{n_3}\right] \le T^{n_3-1}\mathbb{E}_0\left[\int_0^T \sigma^{2n_3}(t, X(t))dt\right] \\
=&T^{n_3-1}\int_0^T \mathbb{E}_0\left[\sigma^{2n_3}(t, X(t))\right]dt \le T^{n_3-1}C_\sigma^{2n_3}\int_0^T \mathbb{E}_0\left[|X(t)|^{2n_3}\right]dt \\
\le&T^{n_3-1}C_\sigma^{2n_3}\int_0^T \mathbb{E}_0\left[\sup_{0\le s\le t}|X(s)|^{2n_3}\right]dt,
\end{aligned} \tag{4.8}
$$

where the second inequality comes from the linear growth property of σ, Assumption 2.1(3). Inequalities (4.6) and (4.8) imply

$$
\mathbb{E}_0\left[\sup_{0\le t\le T}|X(t)|^{2n_3}\right] \le C_{n_3}T^{n_3-1}C_\sigma^{2n_3}\int_0^T \mathbb{E}_0\left[\sup_{0\le s\le t}|X(s)|^{2n_3}\right]dt. \tag{4.9}
$$

Then, by Gronwall inequality (e.g. page 287 of [8] Karatzas and Shreve (1988)),

$$
\mathbb{E}_0\left[\sup_{0\le t\le T}|X(t)|^{2n_3}\right] < \infty. \tag{4.10}
$$

The estimates (4.1), (4.3), (4.6) and (4.10) conclude the inequality (2.29).
To derive uniform integrability of the family (2.30) from (2.29), using Cauchy-Schwartz and Chebyshev's inequalities to get the estimates,

$$
\begin{aligned}
&\sup_{\tau\in\mathscr{T}}\mathbb{E}_0\left[\left(L^{n_1}(\tau;\mu_j)R^{n_2}(\tau;\mu_j)\sup_{0\le t\le T}|X(t)|^{n_3}\right)\mathbb{1}_{\left\{L^{n_1}(\tau;\mu_j)R^{n_2}(\tau;\mu_j)\sup_{0\le t\le T}|X(t)|^{n_3}>A\right\}}\right] \\
\le&\sup_{\tau\in\mathscr{T}}\left(\mathbb{E}_0\left[L^{2n_1}(\tau;\mu_j)R^{2n_2}(\tau;\mu_j)\sup_{0\le t\le T}|X(t)|^{2n_3}\right]\right)^{1/2} \\
&\cdot\left(\mathbb{P}_0\left(L^{2n_1}(\tau;\mu_j)R^{2n_2}(\tau;\mu_j)\sup_{0\le t\le T}|X(t)|^{2n_3} > A\right)\right)^{1/2} \\
\le&\frac{1}{A^{1/2}}\sup_{\tau\in\mathscr{T}}\mathbb{E}_0\left[L^{2n_1}(\tau;\mu_j)R^{2n_2}(\tau;\mu_j)\sup_{0\le t\le T}|X(t)|^{2n_3}\right],
\end{aligned} \tag{4.11}
$$

which, by (2.29), goes to 0 as $A \to \infty$. □

Definition 4.1 *(Viscosity Solutions)*
Let $\mathscr{C}^2\left([0,T]\times\mathbb{R}\times(0,\infty)^{m+1}\times[0,\infty)^m\right)$ *be the set of real-valued twice differentiable functions on* $[0,T]\times\mathbb{R}\times(0,\infty)^{m+1}\times[0,\infty)^m$.

(1) A function $v_0 : [0,T]\times\mathbb{R}\times(0,\infty)^{m+1}\times[0,\infty)^m \to \mathbb{R}$ *is said to be*

(1.1) a viscosity supersolution to the variational inequality (2.50), if for any $(t,x,l,r) \in [0,T]\times\mathbb{R}\times(0,\infty)^{m+1}\times[0,\infty)^m$, *and any function* $\psi \in \mathscr{C}^2\left([0,T]\times\mathbb{R}\times(0,\infty)^{m+1}\times[0,\infty)^m\right)$, *such that*

$$
0 = (v_0-\psi)(t,x,l,r) = \sup_{[0,T]\times\mathbb{R}\times(0,\infty)^{m+1}\times[0,\infty)^m}(v_0-\psi)(t,x,l,r), \tag{4.12}
$$

there is

$$\min\left\{-\frac{\partial}{\partial t}\psi(t,x,l,r)-\mathscr{A}\psi(t,x,l,r)-\alpha(t,x,l,r),v_0(t,x,l,r)+\alpha(t,x,l,r)\right\}\geq 0,\quad (4.13)$$

with terminal condition (2.51);

(1.2) a viscosity subsolution to the variational inequality (2.50), if for any $(t,x,l,r)\in[0,T]\times\mathbb{R}\times(0,\infty)^{m+1}\times[0,\infty)^m$, *and any function* $\psi\in\mathscr{C}^2\left([0,T]\times\mathbb{R}\times(0,\infty)^{m+1}\times[0,\infty)^m\right)$, *such that*

$$0=(v_0-\psi)(t,x,l,r)=\inf_{[0,T]\times\mathbb{R}^n\times\mathbb{R}^{m+1}\times\mathbb{R}^m}(v_0-\psi)(t,x,l,r),\quad (4.14)$$

there is

$$\min\left\{-\frac{\partial}{\partial t}\psi(t,x,l,r)-\mathscr{A}\psi(t,x,l,r)-\alpha(t,x,l,r),v_0(t,x,l,r)+\alpha(t,x,l,r)\right\}\leq 0,\quad (4.15)$$

with terminal condition (2.51);

(1.3) a viscosity solution if it is both a viscosity supersolution and a viscosity subsolution.

(2) The functions $v_0,v_1,\cdots,v_N:[0,T]\times\mathbb{R}\times(0,\infty)^{m+1}\times[0,\infty)^m\to\mathbb{R}$ *are said to be*

(2.1) viscosity supersolutions to the variational inequalities (3.29) and (3.30), if for any $(t,x,l,r)\in[0,T]\times\mathbb{R}\times(0,\infty)^{m+1}\times[0,\infty)^m$, *and any function* $\psi_k\in\mathscr{C}^2\left([0,T]\times\mathbb{R}\times(0,\infty)^{m+1}\times[0,\infty)^m\right)$, *such that*

$$0=(v_k-\psi_k)(t,x,l,r)=\sup_{[0,T]\times\mathbb{R}\times(0,\infty)^{m+1}\times[0,\infty)^m}(v_k-\psi_k)(t,x,l,r),\quad (4.16)$$

for $k=0,1,2,\cdots,N$, *there is*

$$\min\left\{-\left(\frac{\partial}{\partial t}\psi_k-\mathscr{A}\psi_k-\alpha\right)(t,x,l,r),\left(v_k-\mathscr{M}v_{k-1}\right)(t,x,l,r)\right\}\geq 0,\quad (4.17)$$

for $k=1,2,\cdots,N$, *and*

$$\frac{\partial}{\partial t}\psi_0+\mathscr{A}\psi_0+\alpha(t,x,l,r)\geq 0,\quad (4.18)$$

with the boundary condition (3.31);

(2.2) viscosity subsolutions to the variational inequalities (3.29) and (3.30), if for any $(t,x,l,r)\in[0,T]\times\mathbb{R}\times(0,\infty)^{m+1}\times[0,\infty)^m$, *and any function* $\psi_k\in\mathscr{C}^2\left([0,T]\times\mathbb{R}\times(0,\infty)^{m+1}\times[0,\infty)^m\right)$, *such that*

$$0=(v_k-\psi_k)(t,x,l,r)=\inf_{[0,T]\times\mathbb{R}\times(0,\infty)^{m+1}\times[0,\infty)^m}(v_k-\psi_k)(t,x,l,r),\quad (4.19)$$

for $k=0,1,2,\cdots,N$, *there is*

$$\min\left\{-\left(\frac{\partial}{\partial t}\psi_k-\mathscr{A}\psi_k-\alpha\right)(t,x,l,r),\left(v_k-\mathscr{M}v_{k-1}\right)(t,x,l,r)\right\}\leq 0,\quad (4.20)$$

for $k = 1, 2, \cdots, N$, and

$$\frac{\partial}{\partial t}\psi_0 + \mathscr{A}\psi_0 + \alpha(t,x,l,r) \le 0, \tag{4.21}$$

with the boundary condition (3.31);

(2.3) viscosity solutions if they are both viscosity supersolutions and viscosity subsolutions.

Proof of Theorem 2.4. Defining the Hamiltonian function

$$\begin{aligned} H :&[0,T] \times \bar{\mathscr{O}}_n \times \mathbb{R}^{2m+2} \times \mathbb{S}_{(2m+2)} \to \mathbb{R}; \\ &(t,y,p,M) \mapsto H(t,y,p,M), \end{aligned} \tag{4.22}$$

as

$$H(t,y,p,M) = -(\mu(t,y)\cdot p + \text{ trace } \left(\sigma_Y \sigma_Y^{transpose}(t,y)M\right) + \alpha(t,y)), \tag{4.23}$$

where $\mathbb{S}_{(2m+2)}$ is the set of all $(2m+2)\times(2m+2)$ symmetric matrices. The variational inequality (2.64) can be written as

$$\min\left\{-\frac{\partial}{\partial t}v_n(t,y) + H\left(t,y,D_y v_n(t,y), D_y^2 v_n(t,y)\right), v_n(t,y)+\alpha(t,y)\right\} = 0, \ (t,y)\in Q_n\backslash\partial^* Q_n, \tag{4.24}$$

with the boundary condition (2.65). The proof will proceed through four steps.

Step 1. Taking any (t,y^1) and (t,y^2) in Q_n, and any matrices A and B in $\mathbb{S}_{(2m+2)}$ satisfying, for some $\epsilon > 0$

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \le \frac{3}{\epsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \tag{4.25}$$

where I is the $(2m+2)\times(2m+2)$-identity matrix. Denoting $\Sigma_1 = \sigma(t,y^1)$ and $\Sigma_2 = \sigma(t,y^2)$, then from inequality (4.25) and the Lipschitz continuity of $\sigma(t,\cdot)$ over $\mathscr{O}_n$, we deduce that

$$\begin{aligned} &\text{trace } (\Sigma_1\Sigma_1' A - \Sigma_2\Sigma_2' B) = \text{ trace } \left(\begin{pmatrix} \Sigma_2\Sigma_2' & \Sigma_2\Sigma_1' \\ \Sigma_1\Sigma_2' & \Sigma_1\Sigma_1' \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix}\right) \\ \le&\frac{3}{\epsilon}\text{ trace } \left(\begin{pmatrix} \Sigma_1\Sigma_1' & \Sigma_2\Sigma_1' \\ \Sigma_1\Sigma_2' & \Sigma_2\Sigma_2' \end{pmatrix}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}\right) = \frac{3}{\epsilon}\text{ trace } (\Sigma_2\Sigma_2' - \Sigma_2\Sigma_1' - \Sigma_1\Sigma_2' + \Sigma_1\Sigma_1') \\ =&\frac{3}{\epsilon}||\Sigma_2 - \Sigma_1||^2 \le \frac{3}{\epsilon}C_{N,\sigma}||y^1 - y^2||^2. \end{aligned} \tag{4.26}$$

With $p_\epsilon := \frac{1}{\epsilon}(y^1 - y^2)$, from inequality (4.26) and the Lipschitz continuity of $\mu(t,\cdot)$ and $\alpha(t,\cdot)$, there exists a constant $C_{n,H}$, not depending on ϵ, such that

$$H\left(t,y^2,p_\epsilon,B\right) - H\left(t,y^1,p_\epsilon,A\right) \le C_{n,H}\left(\frac{1}{\epsilon}||y^1-y^2||^2 + ||y^1-y^2||\right). \tag{4.27}$$

Step 2. For arbitrary fixed $\epsilon > 0$, $\delta > 0$ and $\eta > 0$, the function Φ defined as

$$\Phi(t,y^1,y^2) = \underline{v}_n(t,y^1) - \bar{v}_n(t,y^2) - \frac{1}{2\epsilon}||y^1-y^2||^2 - \delta(T-t) - \frac{\eta}{t}, \tag{4.28}$$

for $(t, y^1, y^2) \in [0, T] \times \bar{\mathscr{O}}_n \times \bar{\mathscr{O}}_n$, is continuous and has the right limit $-\infty$ as $t \to 0+$. Hence Φ attains its maximum at some point (t_n, y_n^1, y_n^2) in $(0, T] \times \bar{\mathscr{O}}_n \times \bar{\mathscr{O}}_n$. Seeing from

$$\Phi(t_n, y_n^1, y_n^1) + \Phi(t_n, y_n^2, y_n^2) \leq 2\Phi(t_n, y_n^1, y_n^2), \tag{4.29}$$

and from the Lipschitz continuity of $\underline{v}_n$ and $\bar{v}_n$, there exists a constant C_n, not depending on ϵ, such that

$$\begin{aligned} &\frac{1}{2\epsilon}||y_n^1 - y_n^2||^2 \leq \underline{v}_n(t_n, y_n^1) - \underline{v}_n(t_n, y_n^2) + \bar{v}_n(t_n, y_n^1) - \bar{v}_n(t_n, y_n^2) \\ \leq &C_n||y_n^1 - y_n^2|| \leq \epsilon C_n^2 + \frac{1}{4\epsilon}||y_n^1 - y_n^2||^2, \end{aligned} \tag{4.30}$$

which implies that

$$||y_n^1 - y_n^2||^2 \leq 4\epsilon^2 C_n^2. \tag{4.31}$$

Step 3. At least one of the three cases is true: $(t_n, y_n^1) \in \partial^* Q_n$, $(t_n, y_n^2) \in \partial^* Q_n$, or $(t_n, y_n^1), (t_n, y_n^2) \in Q_n$.

Step 3.1. If $(t_n, y_n^1) \in \partial^* Q_n$, then from

$$\Phi(t, y, y) \leq \Phi(t_n, y_n^1, y_n^2), \tag{4.32}$$

by the Lipschitz continuity of $\bar{v}_n$, and by (4.31), there exists a constant $\bar{C}_n$, depending only on $\bar{v}_n$ and $\mathscr{O}_n$, such that

$$\begin{aligned} &\underline{v}_n(t, y) - \bar{v}_n(t, y) \\ \leq &\underline{v}_n(t_n, y_n^1) - \bar{v}_n(t_n, y_n^2) - \frac{1}{2\epsilon}||y_n^1 - y_n^2||^2 + \delta(t_n - t) + \frac{\eta}{t} - \frac{\eta}{t_n} \\ \leq &\sup_{\partial^* Q_n}(\underline{v}_n - \bar{v}_n) + \underline{v}_n(t_n, y_n^2) - \bar{v}_n(t_n, y_n^2) + \delta(T - t) + \frac{\eta}{t} \\ \leq &\sup_{\partial^* Q_n}(\underline{v}_n - \bar{v}_n) + \bar{C}_n||y_n^1 - y_n^2|| + \delta(T - t) + \frac{\eta}{t} \\ \leq &\sup_{\partial^* Q_n}(\underline{v}_n - \bar{v}_n) + 2\epsilon C_n \bar{C}_n + \delta(T - t) + \frac{\eta}{t}, \end{aligned} \tag{4.33}$$

for all (t, y) in $\bar{Q}_n$. In inequalities (4.33), first sending ϵ, ρ and β to zero, then taking supremum of $\underline{v}_n(t, y) - \bar{v}_n(t, y)$ over $\bar{Q}_n$, we obtain the inequality

$$\sup_{\bar{Q}_n}(\underline{v}_n - \bar{v}_n) \leq \sup_{\partial^* Q_n}(\underline{v}_n - \bar{v}_n). \tag{4.34}$$

Step 3.2. If $(t_n, y_n^2) \in \partial^* Q_n$, then inequality (4.34) can be proved by the same type of arguments as in Step 3.1.

Step 3.3. If $(t_n, y_n^1), (t_n, y_n^2) \in Q_n$, then the conditions for the Crandall-Ishii maximum principle (c.f. Theorem V.6.1, Fleming and Soner (1993) [6]) are satisfied. Denoting

$$\phi(t, y^1, y^2) = \frac{1}{2\epsilon}||y^1 - y^2||^2 - \delta(t - T) + \frac{\eta}{t}, \ (t, y^1, y^2) \in [0, T] \times \bar{\mathscr{O}}_n \times \bar{\mathscr{O}}_n. \tag{4.35}$$

There exist q and $\hat{q}$ in $\mathbb{R}$ and matrices A and B in $\mathbb{S}_{(2m+2)}$, satisfying

(1) $q - \hat{q} = D_t\phi(t_n, y_n^1, y_n^2) = -\delta - \frac{\eta}{t_n^2}$;
(2)

$$\begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix} \leq \left(\Sigma_{12} + \epsilon\Sigma_{12}^2\right) = \frac{3}{\epsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}; \tag{4.36}$$

(3) $(q, p, A) \in \bar{\mathscr{P}}^{2,+}\underline{v}_n(t_n, y_n^1)$, $p = D_{y^1}\phi(t_n, y_n^1, y_n^2) = p_\epsilon$;
(4) $(\hat{q}, \hat{p}, B) \in \bar{\mathscr{P}}^{2,-}\bar{v}_n(t_n, y_n^2)$, $\hat{p} = -D_{y^2}\phi(t_n, y_n^1, y_n^2) = p_\epsilon$,

where

$$\Sigma_{12} = D^2_{y^1,y^2}\phi(t_n, y_n^1, y_n^2) = \frac{1}{\epsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \tag{4.37}$$

the set $\bar{\mathscr{P}}^{2,+}\underline{v}_n(t_n, y_n^1)$ is the closure of the parabolic superjet of $\underline{v}_n$ at the point (t_n, y_n^1), and the set $\bar{\mathscr{P}}^{2,-}\bar{v}_n(t_n, y_n^2)$ is the closure of the parabolic subjet of $\bar{v}_n$ at the point (t_n, y_n^2).

Step 4. In the case of Step 3.3, from the equivalent definition of viscosity subsolutions and supersolutions using semijets (Definition 2.2 of Crandall, Ishii and Lions (1992) [5] and Theorem V.4.1 of Fleming and Soner (1993) [6])), the triples (q, p, A) and $(\hat{q}, \hat{p}, B)$ satisfy

$$\min\left\{-q + H\left(t_n, y_n^1, p_\epsilon, A\right), \underline{v}_n(t_n, y_n^1) + \alpha(t_n, y_n^1)\right\} \leq 0, \tag{4.38}$$

and

$$\min\left\{-\hat{q} + H\left(t_n, y_n^2, p_\epsilon, B\right), \bar{v}_n(t_n, y_n^2) + \alpha(t_n, y_n^2)\right\} \geq 0. \tag{4.39}$$

Subtracting (4.39) from (4.38) implies that either

$$\underline{v}_n(t_n, y_n^1) - \bar{v}_n(t_n, y_n^2) \leq -\alpha(t_n, y_n^1) + \alpha(t_n, y_n^2) \tag{4.40}$$

or

$$-q + \hat{q} \leq H\left(t_n, y_n^2, p_\epsilon, B\right) - H\left(t_n, y_n^1, p_\epsilon, A\right) \tag{4.41}$$

is true.

Step 4.1. If inequality (4.40) holds, then by (4.32), (4.31) and the Lipschitz continuity of $\alpha(t, \cdot)$ on the bounded set $\bar{\mathscr{O}}_n$, there exists a constant $C_{n,\alpha}$, not depending on ϵ, such that

$$\begin{aligned} &\underline{v}_n(t, y) - \bar{v}_n(t, y) \\ \leq &\underline{v}_n(t_n, y_n^1) - \bar{v}_n(t_n, y_n^2) + \delta(T - t) + \frac{\eta}{t} \\ \leq &2\epsilon C_n C_{n,\alpha} + \delta(T - t) + \frac{\eta}{t}, \end{aligned} \tag{4.42}$$

for all (t, y) in $\bar{Q}_n$. Hence first letting ϵ, δ and η go to zero, then taking the supremum over $\bar{Q}_n$, there is

$$\sup_{\bar{Q}_n}(\underline{v}_n - \bar{v}_n) \leq 0. \tag{4.43}$$

Step 4.2. If inequality (4.41) holds, then by (4.27), (4.36)-*(1)*, and (4.31), there is

$$\begin{aligned} 0 < \delta \leq & -q + \hat{q} \leq H\left(t, y^2, p_\epsilon, B\right) - H\left(t, y^1, p_\epsilon, A\right) \\ \leq & C_{n,H}\left(\frac{1}{\epsilon}||y^1 - y^2||^2 + ||y^1 - y^2||\right) \leq 2\epsilon C_{n,H} C_n (2C_n + 1). \end{aligned} \tag{4.44}$$

Letting $\epsilon \to 0+$ in (4.44) produces the contradiction that $0 \leq \delta \leq 0$ which simply means that the situation $(t_n, y_n^1), (t_n, y_n^2) \in Q_n$ and the inequality (4.41) are mutually exclusive.

In all cases other than that in Step 4.2, the comparison result (2.70) has been proven. □

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